Let $S$ be a set and let $\circ$ be a binary operation on $S$ satisfying

1. $x \circ x = x$ for all $x \in S$, and
2. $(x \circ y) \circ z = (y \circ z) \circ x$ for all $x, y, z \in S$.

**Prove:** $\circ$ is commutative and associative.

**Proof.** From the given properties, we see that

\[
x \circ y = (x \circ y) \circ (x \circ y) = [y \circ (x \circ y)] \circ x = [(x \circ y) \circ x] \circ y =
\]

\[
= [(y \circ x) \circ x] \circ y = [(x \circ x) \circ y] \circ y = (x \circ y) \circ y = (y \circ y) \circ x = y \circ x.
\]

From this commutative law, we see that

\[
(x \circ y) \circ z = (y \circ z) \circ x = x \circ (y \circ z).
\]

Source: 1971 Putnam Exam.