Problem #499
Solution

The area of \( \triangle ABC \) is 1. Point \( D \) on \( BC \) is one third of the way from \( B \) to \( C \), point \( E \) is one third of the way from \( C \) to \( A \), and point \( F \) is one third of the way from \( A \) to \( B \). Find the area of \( \triangle GHI \).

[Diagram of \( \triangle ABC \) with points \( D, E, F, G, H, I \) labeled]

**Answer:** \( \frac{1}{7} \).

**Proof.** Let \( K_{XYZ} \) denote the area of triangle \( XYZ \). Let \( k = K_{AIF} \). We have \( K_{BIF} = 2K_{AIF} = 2k \) because the triangles have the same height and \( BF = 2AF \). Therefore \( K_{AIB} = 3k \). Similarly, \( K_{CAD} = 2K_{BAD} \). Now \( K_{BAD} = \frac{1}{3}K_{ABC} = \frac{1}{3} \), so

\[
K_{BID} = \frac{1}{3} - K_{AIB} = \frac{1}{3} - 3k.
\]

By the usual reasoning, \( K_{CID} = 2K_{BID} = \frac{2}{3} - 6k \). Therefore
\[
K_{CAI} = \frac{2}{3} - K_{CID} = 6k.
\]
However, \( K_{CAF} = \frac{1}{4}K_{ABC} = \frac{1}{3} \). If follows that \( K_{AIF} = \frac{1}{3} - 6k = k \), so \( K_{AIF} = \frac{1}{21} \). By symmetry,

\[
K_{AIF} = K_{BGD} = K_{CHE} = \frac{1}{21}.
\]

Now \( K_{AIHE} = K_{CAF} - K_{CHE} - K_{AIF} = \frac{1}{3} - \frac{1}{21} - \frac{1}{3} = \frac{5}{21} \). By symmetry,

\[
K_{AIHE} = K_{BGIF} = K_{CHGD} = \frac{5}{21}.
\]

Finally,

\[
K_{GHI} = 1 - (K_{AIF} + K_{BGD} + K_{CHE}) - (K_{AIHE} + K_{BGIF} + K_{CHGD}) = 1 - 3\cdot \frac{1}{21} - 3\cdot \frac{5}{21} = \frac{1}{7}.
\]

Source: This is an oldie but a goodie.