A positive integer is said to *cute* if it can be written as a product of numbers that have only a single digit when written in base ten. For example, $1890 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 9$ and $250 = 2 \cdot 5 \cdot 5 \cdot 5 \cdot 5$ are cute, but 143 is not cute because it is divisible by 11. Note that 11 is prime, so it cannot be written as a product of single digit numbers.

1. 2015 may be written as a sum of two cute numbers because $2015 = 2000 + 15$. Is there another way to write 2015 as a sum of two cute numbers? Find another such representation or else prove that this is impossible.

2. Find a cute number whose last two digits are 57, or else prove that no such number exists.

*Answer:*

1. There are five such representations:

$$2015 = 1890 + 125 = 1875 + 140 = 1715 + 300 = 1280 + 735 = 1215 + 800.$$  

2. No such cute number exists. To see why, observe that any cute number must be a product of the primes 2, 3, 5, and 7. For any number divisible by 2, the last digit will be even. For any number divisible by 5, the last digit must be 0 or 5. Therefore, a cute number with last two digits 57 must be of the form $3^a 7^b$.

Note that the set $\{1, 3, 7, 9\}$ is closed under multiplication mod 20; i.e., the product of any two numbers in this set mod 20 is also a number in this set. Therefore $3^a 7^b$ must be congruent to 1, 3, 7, or 9 mod 20. On the other hand, a number with last two digits 57 is congruent to 17 (mod 20).

Source: Adapted from a problem on the 2014 Michigan Mathematics Prize Competition.