The Fibonacci sequence $F_n$ is defined by the relations

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$ 

Consider the infinite series

$$\sum_{n=0}^{\infty} \frac{F_n}{10^n}.$$ 

Does this series converge? If it converges, is the value of the sum rational? If so, then express the value as a quotient of integers in lowest terms. If the value of the sum is irrational, then give a proof of this fact.

**Answer:** The given series converges to $\frac{10}{89}$.

**Proof.** A straightforward induction argument proves that $F_{n+1}/F_n \leq 2$ for all $n \geq 0$. Therefore the given series converges by the ratio test.

By making changes of variable in the indices of summation, we see that

$$x \sum_{n=0}^{\infty} F_n x^n = \sum_{m=1}^{\infty} F_{m-1} x^m \quad \text{and} \quad x^2 \sum_{n=0}^{\infty} F_n x^n = \sum_{m=2}^{\infty} F_{m-2} x^m.$$ 

Therefore

$$(1 - x - x^2) \sum_{n=0}^{\infty} F_n x^n = F_0 + F_1 x - F_0 x + \sum_{n=2}^{\infty} (F_n - F_{n-1} - F_{n-2}) x^n = x,$$

since $F_n - F_{n-1} - F_{n-2} = 0$ for $n \geq 2$. Therefore,

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$$

for any $x$ that makes the left-hand side converge. Taking $x = 1/10$ gives the answer $10/89$. \qed

Note that taking $x = 1/1000$ gives

$$\sum_{n=0}^{\infty} \frac{F_n}{1000^n} = \frac{1000}{998999} = 0.001 002 003 005 013 021 034 055 089 144 \ldots.$$ 

Source: Allan Schwenk, Western Michigan University