Problem #505
Solution

Note that when \( n = 12 \), \( n \) and \( n + 6 \) have the same set of prime factors because \( 12 = 2^2 \cdot 3 \) and \( 12 + 6 = 2 \cdot 3^2 \). Are there infinitely many positive integers \( n \) such that \( n \) and \( n + 6 \) have the same prime factors? If so, prove it. If not, find the largest integer \( n \) such that \( n \) and \( n + 6 \) have the same prime factors.

**Answer:** The largest \( n \) such that \( n \) and \( n + 6 \) have the same prime factors is 48.

**Proof.** Clearly 48 = \( 2^4 \cdot 3 \) and 48 + 6 = \( 2 \cdot 3^3 \) have the same prime factors. Assume that \( n \) is an integer such that \( n \) and \( n + 6 \) have the same prime divisors. We will show that \( n \) ≤ 48.

Note that if \( p \) is prime dividing \( n \), then \( p \) divides 6. Accordingly, \( p = 2 \) or \( 3 \), and \( n = 2^a3^b \) and \( n + 6 = 2^c3^d \) for non-negative integers \( a, b, c, d \). We distinguish three cases.

1. \( n = 2^a \), with \( a > 0 \).
2. \( n = 3^b \), with \( b > 0 \).
3. \( n = 2^a3^b \), with \( a > 0 \) and \( b > 0 \).

In case 1, \( n + 6 \equiv n \pmod{3} \), so \( n + 6 = 2^c \). Accordingly, \( 2^c - 2^a = 2^a(2^a - c - 1) = 6 = 2 \cdot 3 \). It follows that \( a = 1 \) and \( n = 2 \). In case 2, \( n + 6 \equiv n \pmod{2} \), so \( n + 6 = 3^d \). Accordingly, \( 3^d - 3^b = 3^b(3^d - b - 1) = 6 \). It follows that \( b = 1 \) and \( n = 3 \).

In case 3, \( 2^a3^d - 2^a3^b = 6 \). Writing \( x = c - 1, y = d - 1, a = v - 1, b = w - 1 \), we get

\[ 2^x3^y - 2^v3^w = 1 \]  

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with \( x, y, v, w \) all \( \geq 0 \). Considering this equation mod 2 and mod 3, we see that either \( x = w = 0 \) or \( y = v = 0 \).

If \( y = v = 0 \), then \( 2^x - 3^y = 1 \). If \( x \leq 2 \), then \( n = 2^x13 - 6 \leq 18 < 48 \). If \( x > 2 \), then \( 3^y \equiv 7 \pmod{8} \), which is impossible.

If \( x = w = 0 \), then \( 3^y - 2^v = 1 \). If \( v \leq 3 \), then \( n = 2^v + 3 \leq 48 \). If \( v \geq 4 \), then \( 3^y \equiv 1 \pmod{16} \), and so \( y \equiv 0 \pmod{4} \). Consequently, \( 3^y \equiv 1 \pmod{5} \). However, we then have \( 2^v \equiv 0 \pmod{5} \), which is impossible. \( \square \)

Source: S. W. Graham, J. Holt, and C. Pomerance, Solutions of \( \phi(n) = \phi(n+k) \), *Number Theory in Progress, Proceedings of the International Conference on Number Theory in Honor of the 60th birthday of Andrzej Schinzel*, Walter de Gruyter, 1999, pp. 867-882.