Problem #506  
Solution

Find a closed form evaluation of the integral

$$\int_0^{\infty} \frac{e^{-x} - e^{-2x}}{x} dx.$$

Answer: The integral is log 2. (Here, log denotes the natural logarithm.)

First solution: Write the integral as

$$\int_0^{\infty} \int_x^{2x} e^{-y} \frac{dy}{x} dx.$$

Changing the order of integration yields

$$\int_0^{\infty} \int_{y/2}^{y} e^{-y} \frac{dx}{x} dy = \log 2 \int_0^{\infty} e^{-y} dy = \log 2.$$

Second solution: Let

$$F(a) = \int_0^{\infty} \frac{e^{-x} - e^{-ax}}{x} dx.$$  

Then

$$F'(a) = \int_0^{\infty} e^{-ax} dx = \frac{1}{a},$$

and $F(1) = 0$. Therefore

$$F(2) = \int_1^{2} F'(a) da = \log 2.$$

Source: Adapted from 1982 Putnam Exam