Algebra Qualifying Examination
January 7, 2015

Do all seven (7) problems. Each problem is 10 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

Notation. Throughout the test, we use the following notation:
- \( \mathbb{Z} \) is the ring of integers.
- \( \mathbb{Q} \) is the field of rational numbers.
- \( S_n \) is the permutation group on \( n \) elements.

(1) Let \( G \) be a group. The *commutator subgroup* of \( G \) is the group
\[
G' = \langle xyx^{-1}y^{-1} : x, y \in G \rangle.
\]
Prove each of the following statements.

(a) The set \( G' \) is a normal subgroup of \( G \).
(b) If \( N \) is a normal subgroup of \( G \) and \( G/N \) is abelian, then \( G' \) is a subgroup of \( N \).
(c) If \( H \) is a subgroup of \( G \) and \( G' \subseteq H \), then \( H \) is normal in \( G \).
(d) If \( n \geq 3 \), then the commutator subgroup of \( S_n \) is its alternating group \( A_n \).

(2) Let \( k \geq 1 \) be an integer. Prove that if \( G \) is a group of order \( 2^2 \cdot 3 \cdot 11^k \), then \( G \) is not simple.

(3) Give examples of each of the following. Be sure you justify your assertions.

(a) A ring \( R \) and an ideal \( a \) in \( R \) which is not principal.
(b) A ring \( S \) and a nonzero ideal \( p \) in \( S \) which is prime but not maximal.
(c) An integral domain which is not a UFD, and give an explicit example of nonunique factorization.
(d) A polynomial in \( \mathbb{Q}[X] \) which is irreducible but not Eisenstein at any prime \( p \in \mathbb{Z} \).

(4) (a) Define a Euclidean domain.
(b) Show that \( F[X] \) is Euclidean, where \( F \) is a field. (If you wish to cite the Euclidean Algorithm Theorem, then you must prove it.)

(5) (a) Decompose the polynomial \( f(X) = X^4 - 9X^2 + 14 \) into a product of irreducibles in \( \mathbb{Q}[X] \). Be sure to prove that your irreducible factors are indeed irreducible.
(b) Compute the splitting field \( K \) of \( f(X) \) over \( \mathbb{Q} \).
(c) Compute \( \text{Gal}(K/\mathbb{Q}) \).
(d) Use Galois theory to prove that the polynomial \( X^4 - 18X^2 + 25 \) is irreducible.
(6)  (a) Let $\mathbb{F}_2$ be the field with 2 elements. Define the field of rational functions in the indeterminate $T$ over $\mathbb{F}_2$ by

$$L = \mathbb{F}_2(T) = \left\{ \frac{f(T)}{g(T)} : f(T), g(T) \in \mathbb{F}_2[T], \; g(a) \neq 0 \text{ for some } a \in \mathbb{F}_2 \right\}.$$ 

Prove that $L(\sqrt{\bar{T}})$ is a finite extension of $L$ of degree 2 but is not a Galois extension.

(b) State and prove necessary and sufficient conditions for $\mathbb{Q}(\sqrt{d_1})$ and $\mathbb{Q}(\sqrt{d_2})$ to be isomorphic as field extensions of $\mathbb{Q}$, where $d_1$ and $d_2$ are squarefree in $\mathbb{Q}$.

(7) Let $K = \mathbb{Q}(\sqrt[3]{2} + \omega)$ be a field extension of $\mathbb{Q}$ where $\omega$ is a root of $X^2 + X + 1$.

(a) Prove that $\sqrt[3]{2} \in K$. Deduce that $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$.

(b) Prove that $\text{Gal}(K/\mathbb{Q}) \cong S_3$.

(c) Display the lattice of subgroups of $S_3$. For each subgroup, find a set of corresponding generators in $\text{Gal}(K/\mathbb{Q})$.

(d) For each nontrivial proper subgroup of $\text{Gal}(K/\mathbb{Q})$, find its fixed field.