There are three parts. In the first part you are to do 3 of 4 problems. In the second you are to do 6 of 7 problems. In the third part are to do 3 of 5. Each part is worth 30 points; a grade of 65 out of 90 is expected for passing.

Please do each problem on separate sheets.

In every problem you must give evidence of your reasoning.

The bold face letters $C$, $R$, $Q$, $Z$ stand for the complex, real, rational numbers, and integers respectively.

**Part I** (30 points.) Do three of four. Each problem is worth 10 points. (If you do 4 problems, only the first 3 will be graded.)

1 Let $T$ be a linear transformation from a vector space $V$ to a vector space $W$. Let $B$ denote a basis for $V$. Show that $T(B)$ is a basis for the image of $T$ if and only if $T$ is one-to-one.

2 (a) Define what it means to say that a linear transformation $T$ from a vector space $V$ into $V$ is nilpotent of index $k$.

(b) Assume $T$ is nilpotent of index $k$ on $V$. Show that there is a vector $u \in V$ so that the set $B = \{u, Tu, T^2u, \ldots, T^{k-1}u\}$ is linearly independent.

(c) If $W$ is the linear span of the set $\{u, Tu, T^2u, \ldots, T^{k-1}u\}$ of part (b), find the matrix $A$ which represents the restriction $T_1$ of $T$ to $W$ with respect to the basis $B$.

(d) What would be the matrix $A$ in part (c) if we wrote the basis in the order $\{T^{k-1}u, T^{k-2}u, \ldots, Tu, u\}$?

3 (a) Prove that if $A$ and $B$ are similar matrices, then $A$ and $B$ have the same characteristic polynomials and the same minimal polynomials.

(b) Give an example of two non-similar matrices $A$, $B$ with minimal polynomial $m(x) = (x + 3)^2(x - 5)^3$ and characteristic polynomial $c(x) = (x + 3)^4(x - 5)^3$. If this is not possible, explain why.

4 Let $V$ be a vector space with inner product $<, >$ and let $S$ be a subspace of $V$. The orthogonal complement $S^\perp$ is defined as the set $\{x \in V : <x, s> = 0 \ \forall s \in S\}$.

(a) Show that the orthogonal complement of a subspace is a subspace.

(b) Let $S$ be a subspace of $C^4$ with orthonormal basis $B = \{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}), (\frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{1}{2})\}$. The vector $u = (3, 25, 79, 101)$ is a member of $S$. Write $u$ as a linear combination of the vectors in $B$.

(c) Compute $S^\perp$ where $V$ and $S$ are from part (b).

(d) Prove in general that if $V$ is a vector space with inner product $<, >$ and subspace $S$ then $V = S \oplus S^\perp$.
Part II (30 points.) Do six of seven. Each problem is worth 5 points. (If you do 7 problems, only the first 6 will be graded.)

1. (a) How many Sylow 5-subgroups does $S_5$ (the symmetric group on 5 symbols) have?
   (b) List the elements in one of them.

2. State and prove Cayley’s theorem.

3. (a) Give an example of an abelian simple group.
   (b) Give an example of a nonabelian simple group.

4. Let $M_{nn}$ denote the set of all $n \times n$ matrices with real number entries. Name an operation $*$ on $M_{nn}$, a group $(S, *)$ where $S$ is a subset of $M_{nn}$ containing more than one matrix, and a nontrivial group $(G, \circ)$ so that
   (a) The determinant function is a homomorphism from $(S, *)$ onto $(G, \circ)$.
   (b) The transpose function $(A \rightarrow A')$ is a homomorphism from $(S, *)$ onto $(G, \circ)$.
   (Any of $S, *, G$, and $\circ$ may be different in (b) than those you created in (a)).

5. (a) Describe all elements of the group $\mathbb{Q}/\mathbb{Z}$ that have finite order.
   (b) Describe all elements of the group $\mathbb{R}/\mathbb{Q}$ that have finite order.

6. Let $G$ be a group of order 95 and suppose $f$ is a homomorphism from $G$ onto a group $H$.
   Suppose also that $f$ is not an isomorphism. Prove that $H$ is cyclic.

7. A group is solvable iff there exist subgroups

   \[ G = N_0 \supseteq N_1 \supseteq N_2 \supseteq \ldots \supseteq N_r = \{e\} \]

   such that each $N_i$ is normal in $N_{i-1}$ and $N_i/N_{i-1}$ is abelian. Prove that a subgroup of a solvable group is solvable.
Part III (30 points.) Do three of five. Each problem is worth 10 points. (If you do more than three problems, only the first 3 will be graded.)

1 Set \( \Psi_p(x) := \frac{\frac{p^2 - 1}{x}}{x - 1} \) where \( p \) is a prime.
   (a) Prove using Eisenstein’s criterion that \( \Psi_p(x) \) is irreducible over \( \mathbb{Q} \).
   (b) Is \( \Psi_p(x) \) is irreducible over \( \mathbb{Z} \)?
   (c) Prove or disprove: If \( a(x) + \frac{\Psi_p(x)}{< \Psi_p(x)>} \) is a nonzero member of the quotient ring \( \mathbb{Q}[x]/<\Psi_p(x)> \) then \( a(x) + \frac{\Psi_p(x)}{<\Psi_p(x)>} \) is a unit.

2 Let \( G' \) represent the derived subgroup of a group \( G \), that is, the group generated by the commutators of \( G \).
   (a) Prove that if \( N \) is a normal subgroup of \( G \) such that \( G/N \) is abelian then \( G' \subseteq N \).
   (b) Prove that if \( \phi \) is a homomorphism into the multiplicative group of nonzero complex numbers then the kernel of \( \phi \) contains \( G' \).
   (c) Compute \( G' \) when \( G \) is the dihedral group \( < a, b : a^4 = b^2 = 1, bab^{-1} = a^3 > \) of order eight and classify the abelian group \( G/G' \).

3 Let \( \zeta = e^{\frac{2\pi i}{15}} \) so that \( \zeta^{13} = 1 \). Let \( \theta = 1 + \zeta^3 + \zeta^9 \).
   (a) Explicitly describe the Galois group \( G = \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \) and its subgroups.
   (b) Identify the subgroup of \( G \) which fixes \( \mathbb{Q}(\zeta) \).
   (c) Find the dimension of \( \mathbb{Q}(\theta) \) over \( \mathbb{Q} \) and give a basis for \( \mathbb{Q}(\theta) \) over \( \mathbb{Q} \).
   (d) Construct the irreducible polynomial of \( \theta \) over \( \mathbb{Q} \).

4 (a) Let \( f(x) \) and \( g(x) \) be polynomials over an integral domain \( D \). Suppose \( \text{GCD}(f(x), g(x)) = h(x) \). Prove the following version of the Chinese Remainder Theorem:
   The pair of simultaneous congruences
   \[
   t(x) \equiv a(x) \mod f(x) \\
   t(x) \equiv b(x) \mod g(x)
   \]
   has a solution modulo \( \frac{f(x)g(x)}{h(x)} \) if and only if \( h(x) \) divides \( (a(x) - b(x)) \).
   (b) Solve the following pair of simultaneous congruences over the polynomial ring \( \mathbb{Q}[x] \):
   \[
   t(x) \equiv x^4 \mod (x^4 + 2x^3 - 3x - 6) \\
   t(x) \equiv x^2 - 10x - 8 \mod (x^3 + 2x^2 + x + 2)
   \]

5 For this problem, \( F \) is the field of order 3. You are given that \( f \) and \( g \) are irreducible polynomials over \( F \) of degrees 4 and 2 respectively. Set \( R := F[x]/<f \cdot g> \).
   (a) Prove or disprove \( R \) is a field.
   (b) Prove or disprove \( R \) is an integral domain.
   (c) Find the cardinality of \( R \).
   (d) Classify the group of units of \( R \) as a direct product of abelian groups.