Do all six problems. The exam is 50 points. A passing grade is 35/50.
No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. (10 points)
   Let $k \geq 1$ be an integer. Prove: No group of order $2^k \cdot 5$ is simple. (Note: If you use Burnside’s Theorem, you should prove it. Alternately, if you use the Classification of Finite Simple Groups, you should prove it.)

2. (5 points)
   Let $F$ be a field and let $f(x) \in F[x]$ be an irreducible polynomial. Suppose that $E$ is a splitting field for $F$ and assume that there exists an element $\alpha \in E$ such that both $\alpha$ and $\alpha + 1$ are roots of $f(x)$. Show that the characteristic of $F$ is not zero.

3. (5 points)
   Let $X$ be a subspace of $M_n(\mathbb{C})$, the $\mathbb{C}$-vector space of all $n \times n$ complex matrices. Assume that every nonzero matrix in $X$ is invertible. Prove that $\dim_{\mathbb{C}} X \leq 1$.

4. (10 points)
   Let $V$ be a finite-dimensional vector space over an algebraically closed field $F$, and let $S$ and $T$ be commuting linear operators on $V$. Assume that the characteristic polynomial of $S$ has distinct roots.
   (a) (5 points) Prove: Every eigenvector of $S$ is an eigenvector of $T$.
   (b) (5 points) Prove: If $T$ is nilpotent, then $T = 0$.

5. (10 points)
   Let $Q$ be the field of rational numbers, and let $f(x) = x^8 + x^4 + 1$ be a polynomial in $Q[x]$. Suppose $F$ is a splitting field for $f(x)$ over $Q$ and set $G = \text{Aut}_Q F$.
   (a) (5 points) Find $[F : Q]$, and determine the Galois group $G$ up to isomorphism.
   (b) (5 points) If $\Omega \subseteq F$ is the set of roots of $f(x)$, find the number of orbits for the action of $F$ on $\Omega$. 
6. (10 points)
Let $M$ be a $\mathbb{Z}$-module; i.e., $M$ is an abelian group. Suppose that

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

is a chain of submodules such that, for $i = 1, 2, \ldots, n$, the factors $M_i/M_{i-1}$ are simple and pairwise non-isomorphic.

**Prove:** If $X$ and $Y$ are isomorphic submodules of $M$, then $X = Y$. 