Algebra Qualifying Exam  
January 5, 2010

Do all seven problems. The exam is 70 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Let $G$ be a group of order $132 = 2^2 \cdot 3 \cdot 11$. Prove that $G$ has a normal $p$-Sylow subgroup for some prime $p$ that divides 132.

2. (a) Let $S_1$ and $S_2$ be commutative rings with 1. Consider the product ring $S_1 \times S_2$ with binary operations defined componentwise. Prove that $(a, b)$ is a unit in $S_1 \times S_2$ if and only if $a$ and $b$ are units in $S_1$ and $S_2$ respectively.

(b) Let $p$ be a prime in $\mathbb{Z}$. Assume $d_1$ and $d_2$ are positive integers and $d_1 < d_2$. Consider the natural ring projection $\varphi : \mathbb{Z}/p^{d_2}\mathbb{Z} \to \mathbb{Z}/p^{d_1}\mathbb{Z}$; that is for any $\overline{m} \in \mathbb{Z}/p^{d_2}\mathbb{Z}$, $\varphi(\overline{m}) \equiv m \pmod{p^{d_1}}$. Prove that if $\overline{m} \in (\mathbb{Z}/p^{d_2}\mathbb{Z})^\times$, then $\varphi(\overline{m}) \in (\mathbb{Z}/p^{d_1}\mathbb{Z})^\times$. Thus $\varphi$ induces a well-defined group homomorphism $\eta : (\mathbb{Z}/p^{d_2}\mathbb{Z})^\times \to (\mathbb{Z}/p^{d_1}\mathbb{Z})^\times$.

(c) Prove that the induced homomorphism

$$\eta : (\mathbb{Z}/p^{d_2}\mathbb{Z})^\times \to (\mathbb{Z}/p^{d_1}\mathbb{Z})^\times$$

is surjective.

(d) Prove that the natural surjective ring projection $\mathbb{Z}/500\mathbb{Z} \to \mathbb{Z}/25\mathbb{Z}$ induces a surjective homomorphism on the group of units

$$(\mathbb{Z}/500\mathbb{Z})^\times \to (\mathbb{Z}/25\mathbb{Z})^\times.$$  

(Hint: Use the Chinese Remainder Theorem and the previous parts.)
3. Let $G$ be a group, and let $Z(G)$ be the center of $G$.

(a) Let $a \in G$. An inner automorphism of $G$ is a function of the form $\gamma_a : G \rightarrow G$ given by $\gamma_a(g) = aga^{-1}$. Let $\text{Inn}(G)$ be the set of all inner automorphisms of $G$. **Prove:** $\text{Inn}(G) \cong G/Z(G)$.

(b) Let $\phi$ be an automorphism of $S_3$. Show that $\phi$ permutes the set $\{(12), (13), (23)\}$, and no non-trivial automorphism of $S_3$ leaves all three elements of this set fixed. Deduce that all automorphisms of $S_3$ are inner automorphisms.

4. The splitting field $E$ of $x^4 + 1$ over $\mathbb{Q}$ is a a simple extension. Find a primitive element for $E$ and determine $\text{Gal}(E/\mathbb{Q})$.

5. Suppose that $q = p^k$ for some positive integer $k$ and some prime $p$. Let $\mathbb{F}_q$ denote the finite field with $q$ elements.

(a) Prove that $x^{p^k} - x$ is a separable polynomial over $\mathbb{F}_p$. Prove that every element of $\mathbb{F}_q$ is a root of $x^{p^k} - x$.

(b) What is the isomorphism type of $\mathbb{F}_q^\times$ as an abelian group? Explain.

(c) Prove that the equation $a^3 = 1$ has 3 solutions in $\mathbb{F}_q$ if and only if $q \equiv 1 \pmod{3}$.

6. Let $p$ be an odd prime, and consider the group $S_{2p}$.

(a) Let $H$ be a $p$-Sylow subgroup of $S_{2p}$. Prove that $H$ has order $p^2$, find the isomorphism type of $H$, and give generators for $H$.

(b) How many $p$-Sylow subgroups does $S_{2p}$ have?

7. Let $R$ be a commutative ring with 1 and let $I$ and $J$ be ideals of $R$. Assume also that neither $I$ nor $J$ is the zero ideal and that neither $I$ nor $J$ contains 1. Let $p$ be a prime ideal in $R$ containing $IJ$.

(a) Prove that $p$ contains either $I$ or $J$.

(b) If $I$ and $J$ are comaximal (i.e. $I + J = R$), then $IJ$ is properly contained in $p$. 