Algebra Qualifying Exam
January 5, 2011

Do all 7 problems. The exam is 70 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Let $G$ be a group of order $pq$ where $p$ and $q$ are primes with $p < q$.
   (a) Prove that $G$ is not a simple group.
   (b) Let $P$ be a Sylow $p$-subgroup. Prove that $G$ is cyclic if and only if $P$ is normal in $G$.

2. (a) Prove that $\mathbb{F}_{11}[x]/(x^2 + 1)$ is a field consisting of 121 elements.
    (b) Construct a field of 81 elements.

3. Let $R$ be a Euclidean Domain. Let $m$ be the minimum integer in the set of norms of nonzero elements of $R$. Prove that every nonzero element of $R$ of norm $m$ is a unit. Deduce that a nonzero element of norm zero (if such an element exists) is a unit.

4. Suppose $n$ is an integer with $n \geq 3$. Recall that the dihedral group $D_{2n}$ is the group with the presentation

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1, s^{-1}rs = r^{-1} \rangle.$$ 

Prove that the center of $D_{2n}$ is $\{1\}$ if $n$ is odd and $\{1, r^{n/2}\}$ if $n$ is even.

5. Let $f(x) = x^4 - 12x^2 + 25$.
   (a) Observe that $f(x + 1) = x^4 + 4x^3 - 6x^2 - 20x + 14$. Using this or otherwise, prove that $f$ is irreducible over $\mathbb{Q}$.
   (b) Determine the Galois group $G$ of the splitting field of $f(x) = x^4 - 12x^2 + 25$ over $\mathbb{Q}$. Express each element of $G$ as a permutation of the roots of $f$.

Continued on the other side.
6. Let $R$ be a commutative ring with unity. We say that $R$ is a *local ring* if $R$ has a unique maximal ideal.

(a) Prove that if $R$ is a local ring with maximal ideal $M$, then every element of $R - M$ is a unit.

(b) Conversely, prove that if the set of nonunits of $R$ forms a proper ideal $M$; namely, $M \neq R$, then $R$ is a local ring.

7. (a) **Prove:** If $G$ is a group and $|G| = 2^2 \cdot 5 \cdot 19$, then $G$ has a normal Sylow $p$-subgroup for either $p = 5$ or $p = 19$.

(b) Let $K$ be a Galois extension of $\mathbb{Q}$. **Prove:** If $[K : \mathbb{Q}] = 2^2 \cdot 5 \cdot 19$, then there exists some subfield $E$ of $K$ such that $E$ is Galois over $\mathbb{Q}$ and $[K : E]$ is prime.