Algebra Qualifying Exam
August, 2009

Do all seven problems. The exam is 70 points. No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you but not in a way that makes it trivial.

1. Prove that $(\mathbb{Z}/32\mathbb{Z})^\times$ is not cyclic.

2. Let $D_{16}$ be the dihedral group of order 16 and let $s$ and $r$ be the commonly used generators for reflection and rotation respectively. Let $H$ denote the subgroup generated by $r^4$ and $sr^2$.
   (a) Determine the isomorphism type of $H$.
   (b) It is known that $D_{16}$ has three subgroups of order 8:

$$\langle s, r^2 \rangle, \langle r \rangle, \langle sr, r^2 \rangle$$

Determine the centralizer $C_{D_{16}}(H)$ and the normalizer $N_{D_{16}}(H)$.

3. (a) Prove that no group of order $84 = 2^2 \cdot 3 \cdot 7$ is simple.

(b) Let $G$ be a group of order $2^k \cdot 3 \cdot 7$. Follow steps i through iii to prove that if $k \geq 19$, then no group of order $2^k \cdot 3 \cdot 7$ is simple.

We use $\mathcal{P}$ to denote the set of all Sylow 2-subgroups of $G$.

i. Describe the number of elements in $\mathcal{P}$.

ii. Prove that there exists a group homomorphism $\varphi : G \rightarrow S_{21}$ induced by conjugating elements in $\mathcal{P}$.

iii. It is a fact that $2^{19} \nmid 21!$. (You may assume this without proof.) Prove that if $k \geq 19$, then the group homomorphism $\varphi$ is not injective. Deduce that $G$ is not a simple group.
4. Let \( x^3 - 2x + 1 \) be an element of the polynomial ring \( E = \mathbb{Z}[x] \) and use the bar notation to denote passage to the quotient ring \( \mathbb{Z}[x]/(x^3 - 2x + 1) \). Let \( p(x) = x^3 + 2x^2 - 1 \) and let \( q(x) = (x - 1)^4 \).

   (a) Express each of \( \overline{p(x)} + \overline{q(x)} \) and \( \overline{p(x)} \overline{q(x)} \) in the form of \( \overline{f(x)} \) for some polynomial \( f(x) \) of degree \( \leq 2 \).

   (b) Prove that \( \overline{E} \) is not an integral domain.

   (c) Prove that \( \overline{x} \) is a unit in \( \overline{E} \).

5. Prove that a finite integral domain is a field. Deduce that if \( R \) is a finite commutative ring with identity, then every prime ideal of \( R \) is a maximal ideal.

6. (a) Determine the splitting field of \( x^3 - 2 \) over \( \mathbb{Q} \), denoted \( E \).

   (b) Let \( G \) be the Galois group of \( E \) over \( \mathbb{Q} \). For each subgroup of \( G \), including \( G \) itself, determine its corresponding fixed field.

7. (a) Find the minimal polynomial of \( \sqrt{5} + 2\sqrt{6} \) over \( \mathbb{Q} \). Determine the degree of the extension field \( \mathbb{Q}(\sqrt{5} + 2\sqrt{6}) \) over \( \mathbb{Q} \). (You must address the irreducibility of the polynomial.)

   (b) Prove that the extension in part (a) is a Galois extension, and determine the isomorphism type of its Galois group.