1. (9 points)
Let $k \geq 1$ be an integer, and let $p$ be an odd prime. Prove: No group of order $8p^k$ is simple. (Note: If you use Burnside’s Theorem, you should prove it. Alternately, if you use the Classification of Finite Simple Groups, you should prove it.)

2. (6 points)
Suppose that $G$ is a group of order 12 and that $P$ is a 3-Sylow subgroup of $G$.
   (a) Prove: If $P$ is normal in $G$ and $P = \langle x \rangle$, then $|C_G(x)| \geq 6$.
   (b) Prove: If $P$ is normal in $G$, then $Z(G)$ has an element of order 2.
   (c) Prove: If $Z(G)$ has no element of order 2, then $G \cong A_4$.

Recall that a ring $R$ satisfies the maximal condition on ideals if every non-empty subset of ideals of $R$ contains a maximal element with respect to inclusion.

3. (10 points)
Let $R$ be an integral domain and write $(a)$ for the principal ideal generated by $a \in R$. Recall that an element of $R$ is said to be irreducible if it is nonzero, not a unit, and has no proper factorization.
   i. Show that $(a) \subseteq (b)$ if and only if $b|a$, and that $(a) = (b)$ if and only if $b = au$ for some unit $u \in R$. (2 points)
   ii. If $R$ is a UFD (unique factorization domain), prove that the set of principal ideals of $R$ satisfies the maximal condition. (4 points)
   iii. If the set of principal ideals of $R$ satisfies the maximal condition, show that every nonzero, nonunit element of $R$ can be written as a finite product of irreducible elements. (4 points)

Continued on the other side.
4. (5 points) Let $F$ be a field and let $f(x)$ be an irreducible polynomial in $F[x]$. Show that if $K$ is a Galois extension of $F$, then all irreducible factors of $f(x)$ in $K[x]$ have the same degree.

5. (5 points) Find the Galois group of $f(x) = x^4 - x^2 + 1$ over $\mathbb{Q}$.

6. (15 points) Find the characteristic polynomial, the minimal polynomial, and the Jordan canonical form for the real matrix $A$, given below. Find a basis of $\mathbb{R}^3$ relative to which $A$ is in its Jordan canonical form.

$$A = \begin{bmatrix} 2 & 6 & -15 \\ 1 & 1 & -5 \\ 1 & 2 & -6 \end{bmatrix}.$$