1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $f^{-1}(E)$ is a Borel set whenever $E$ is a Borel set. Explain why this shows that $f$ is a measurable function.

2. Let $\langle f_n \rangle$ be a sequence of nonnegative functions. Show

$$\int \lim f_n \leq \lim \int f_n.$$

3. Let $\langle f_n \rangle$ be a sequence of measurable functions such that $f_n \to 0$ in measure, and suppose that for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} m(\{x : |f_n(x)| > \epsilon\}) < \infty.$$

Show that $f_n \to 0$ a.e.

Hint: Consider the sets $A(\epsilon) = \{x : \forall k, \exists n \geq k, |f_n(x)| > \epsilon\}$ and $A_n(\epsilon) = \{x : |f_n(x)| > \epsilon\}$.

4. Suppose $f$ is an integrable function on $\mathbb{R}$ then

$$\lim_{t \to 0} \int_{-\infty}^{\infty} |f(x + t) - f(x)| \, dx = 0.$$

5. (a) The Fundamental Theorem of Calculus tells us that if $f$ is continuous on $[a, b]$ then

$$\frac{d}{dx} \int_{a}^{x} f(t) \, dt = f(x).$$

Does this still hold if $f$ is simply integrable? If yes prove your answer. If not, what conditions on $f$ are necessary for it to hold.

(b) The Fundamental Theorem of Calculus also tells us that if $f'(x)$ is continuous then

$$\int_{a}^{x} f'(t) \, dt = f(x) - f(a).$$

Is there a wider class of functions for which this holds? Prove your answer.

6. Let a sequence $\langle g_n \rangle$ in $L^q[0, 1]$, $1 < q < \infty$ have the property that $\left| \int_{0}^{1} f g_n \right| \leq ||f||_p$ for all $n$ and all $f \in L^p[0, 1]$, $\frac{1}{p} + \frac{1}{q} = 1$. Answer the following questions with a proof for a yes answer, and a counterexample for a no answer.

(a) Does it follow that $\langle ||g||_q \rangle$ is bounded?

(b) Must there be a subsequence $\langle g_{n_k} \rangle$ of $\langle g_n \rangle$ and a $g \in L^q[0, 1]$ such that

$$||g_{n_k} - g||_q \to 0$$

as $k \to \infty$?
MTH 636: Provide complete solutions to 6 of the 7 questions.

1. Let \( f(z) = \begin{cases} 
    z^5 & \text{if } z \neq 0 \\
    \frac{1}{|z|^4} & \text{if } z = 0
\end{cases} \)

   Show that the Cauchy-Riemann equations hold at \( z = 0 \), but \( f \) is not differentiable at \( z = 0 \).

2. If \( f \) is analytic in the annulus \( 1 \leq |z| \leq 2 \) and \( |f(z)| \leq 3 \) on \( |z| = 1 \) and \( |f(z)| \leq 12 \) on \( |z| = 12 \), prove that \( |f(z)| \leq 3|z|^2 \) for \( 1 \leq |z| \leq 2 \).
   Hint: Consider \( \frac{f(z)}{z^2} \).

3. Let \( g \) be a continuous on the real interval \([-1, 2]\), and for each complex number \( z \) define

   \[ F(z) := \int_{-1}^{2} g(t) \sin(zt) \, dt. \]

   Prove that \( F \) is entire, and find its power series around the origin. Also, prove that for all \( z \)

   \[ F'(z) := \int_{-1}^{2} t g(t) \cos(zt) \, dt. \]

4. Find the Laurent series for the function

   \[ f(z) = \frac{1}{(z-1)(z-2)} \]

   in each of the following domains:

   (a) \( |z| < 1 \)  (b) \( 1 < |z| < 2 \)  (c) \( 2 < |z| \)

5. Does there exist a function \( f(z) \) analytic in \( |z| < 1 \) and satisfying

   \[ f \left( \frac{1}{2} \right) = \frac{1}{2}, \quad f \left( \frac{1}{3} \right) = \frac{1}{2}, \quad f \left( \frac{1}{4} \right) = \frac{1}{4}, \quad f \left( \frac{1}{5} \right) = \frac{1}{4}, \quad \ldots, \]

   \[ f \left( \frac{1}{2n} \right) = \frac{1}{2n}, \quad f \left( \frac{1}{2n+1} \right) = \frac{1}{2}, \quad \ldots \]?

   Justify your answer.

6. Prove that the equation \( z + 3 + 2e^z = 0 \) has precisely one root in the left half-plane.

7. Using the method of residues verify that

   \[ \int_{-\pi}^{\pi} \frac{1}{1 + \sin^2 \theta} \, d\theta = \pi \sqrt{2}. \]