Analysis Qualifying Exam

Section A  Answer 10 of the 12.

1. Prove that a sequence \( \langle \mathbf{x}_m \rangle \) in \( \mathbb{R}^n \) converges to \( \mathbf{x} \) in \( \mathbb{R}^n \) if and only if \( x_{mj} \rightarrow x_j \) for each \( j = 1, 2, \ldots, n \) where \( \mathbf{x}_m = (x_{m1}, x_{m2}, \ldots, x_{mn}) \) and \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \).

2. Prove the continuity of the following function. State carefully the theorem(s) used.

\[
f(x) = \sum_{n=0}^{\infty} \frac{\sin nx}{1 + n^2}, \quad \text{for } x \text{ real.}
\]

3. Is \( f(x, y) = \begin{cases} \frac{x(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \) differentiable? Prove your answer.

4. (a) The Mean Value Theorem for Integrals for functions of one variable states that if \( f \) is continuous on \([a, b] \), then there exists a point \( x_0 \in [a, b] \) such that \( \int_a^b f(x) \, dx = f(x_0)(b - a) \).

Is there a parallel theorem for functions defined on a compact convex subset of \( \mathbb{R}^n \)? If your answer is yes state and prove the result. If no, give an example to explain why such a result can not be valid.

(b) What conditions on the partial derivatives at a point imply that the point is a saddle point?

5. For each \( n \) let \( f_n(x) = x^n \), for \( x \in [0, 1] \). Prove that \( f_n(x) \rightarrow f(x) \) for all \( x \in [0, 1] \) where \( f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases} \)

Is the convergence uniform? Prove your answer. Does \( \int_0^1 f_n(x) \, dx \rightarrow \int_0^1 f(x) \, dx \)? Does this follow from any of the theorems you have learned? If yes, state the theorem. Or does this contradict any theorems?

6. Let \( E \) be a countable subset of \( \mathbb{R} \).

(a) Using the definition of outer measure prove that \( m^* E = 0 \).

(b) Is \( E \) measurable and what is its measure? Prove your answer.

7. Let \( f \) be an integrable function on \( \mathbb{R} \).

(a) Prove that there exists a set \( E_\infty \) of measure zero such that \( f \) is finite for all \( x \notin E_\infty \).

(b) Prove that there exists a sequence \( \langle f_n \rangle \) of bounded measurable functions, each vanishing outside a set of finite measure such that \( f_n \rightarrow f \) a.e.

8. If \( F \) is absolutely continuous on \([0, 1]\) and \( F'(x) = \frac{1}{\sqrt{x}} \) a.e., how would you compute \( F(x) \) for \( x \in [0, 1] \)? Justify your arguments.
9. Give an example, if possible, of each of the following. Prove your answer.

(a) A function \( f \) on \([0, \infty)\) such that \( \int_0^\infty f(x) \, dx \) exists as an improper Riemann integral but \( f \) is not Lebesgue integrable on \([0, \infty)\).
(b) A Lebesgue integrable function that is not Riemann integrable.
(c) A sequence \( \{f_n\} \) of nonnegative integrable functions defined on \([0, 1]\) such that \( \int_{[0,1]} f_n \to 0 \), but the sequence does not converge.

10. Find the number of solutions of \( f(z) = (z^6 + 2z - 1) - 5z^3 = 0 \) interior to the unit circle.

11. (a) State Fatou’s Lemma.
(b) State the Dominated Convergence Theorem, and use Fatou’s Lemma to prove it.

12. Let \( f(z) \) be an entire function such that \( |f(z)| \leq 1 + |z|^2 \) for all complex numbers \( z \). Prove that \( f(z) \) is a polynomial.

**Section B**  Select one of the two parts

**Part 1**  Answer 2 of the three

1. Let \( \mu \) denote Lebesgue measure on \([0, 1]\) and \( f \) a nonnegative integrable function on \([0, 1]\).
   (a) Define the set function \( \nu \) by \( \nu(E) = \int_E f \, d\mu \) where \( E \) is a measurable subset of \([0, 1]\). Show \( \nu \) is a measure on \([0, 1]\).
   (b) Show \( \int_E g \, d\nu = \int_E gf \, d\mu \).

2. Let \((X, M, \mu)\) be a measure space and let \( f : X \to \mathbb{R} \) be integrable. Prove Chebyshev’s inequality: for every \( \lambda > 0 \)
\[ \mu \left( \{x \in X : |f(x)| > \lambda\} \right) \leq \frac{1}{\lambda} \int_X |f| \, d\mu. \]

3. (a) Let \( f \) be a function such that all the indicated norms are defined. What are \( \|f\|_p \), for \( 1 \leq p \leq \infty \).
   (b) Discuss the containments among the \( L^p \), \( 1 \leq p \leq \infty \) spaces over a finite measure space. Justify your answer.
   (c) Do part (b) for an infinite measure space.

**Part 2**  Answer 2 of 3

1. Find a conformal mapping of the upper half plane, \( \text{Im} \ z > 0 \), onto the disk \( |w| < R \) such that a given point \( \alpha \) in the upper half plane is mapped to the center of the disk. Is your answer unique? Explain.

2. Find the function \( u(x, y) \) that is harmonic in the unit disk \( |z| < 1 \), and takes on the boundary values \( u(\theta) = u(\cos \theta, \sin \theta) = \theta / 2 \) for \( -\pi < \theta < \pi \).

3. Prove: if the radius of convergence of the series \( f(z) = \sum a_n z^n \) is \( R \), then \( f(z) \) has at least one singular point on the circle \(|z| = R|\).