Analysis Qualifying Exam: September 4, 2009

MTH 632: Provide complete solutions to 5 of the 6 questions.

Notation: \( \mathbb{Q} \) denotes the set of rational numbers and \( \mathbb{R} \) denotes the set of real numbers.

1. Assume \( A, B, G \subset \mathbb{R} \) and \( m^* \) is Lebesgue outer measure on \( \mathbb{R} \). Here \( \tilde{G} \) denotes the complement of \( G \).

   (a) Suppose \( G \) is measurable and \( A \subset G \) and let \( B \) be such that \( B \cap G = \emptyset \). Show \( m^*(A \cup B) = m^*(A) + m^*(B) \).

   (b) Suppose \( A \) and \( B \) are such that \( \text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\} > 0 \). Show \( m^*(A \cup B) = m^*(A) + m^*(B) \).

2. Let \( f(x) = \begin{cases} x(1-x) & ; \quad x \in [0,1] \setminus \mathbb{Q} \\ 1 & ; \quad x \in [0,1] \cap \mathbb{Q} \end{cases} \). Find \( \int_{[0,1]} f \, dm \). Is \( f \) Riemann integrable? Explain your answer.

3. Let \( f \in L^1(\mathbb{R}) \). Show there is a sequence \( \{x_n\} \subset \mathbb{R} \) with \( \lim_{n \to \infty} x_n = \infty \) such that \( \lim_{n \to \infty} x_n f(x_n) = 0 \).

4. Let \( g \) be a function defined on \( \mathbb{R} \) such that there is a constant \( \lambda > 0 \) such that

\[
|g(x) - g(y)| \leq \lambda |x - y|, \quad \forall x, y \in \mathbb{R},
\]

i.e. \( g \) satisfies a Lipschitz condition on \( \mathbb{R} \) and is hence continuous. Let \( f \in L^1([a, b]) \). Show \( g \circ f \), the composition of \( f \) and \( g \) is Lebesgue integrable on \( [a, b] \).

5. Let \( \{f_n\} \) be a sequence of integrable functions defined on a measurable set \( E \subset \mathbb{R} \). The sequence \( \{f_n\} \) is said to be equi-integrable on \( E \) if \( \forall \epsilon > 0, \exists \delta > 0 \) such that \( \forall \) measurable sets \( A \subset E \) with \( m(A) < \delta \) we have \( \int_A |f_n| \, dm < \epsilon, \forall \ n \). Suppose \( \{f_n\} \) is a convergent sequence, say \( f_n \to f \), of equi-integrable functions on a measurable set \( E \), \( m(E) < \infty \). Then \( \lim_{n \to \infty} \int_E f_n \, dm = \int_E f \, dm \).

6. Let \( \{f_n\} \) be a sequence of nonnegative measurable functions on a set \( E \) such that \( \lim_{n \to \infty} \int_E f_n \, dm = 0 \). Show \( \{f_n\} \) converges to zero in measure. Show convergence in measure cannot be replaced with convergence almost everywhere.
MTH 636: Provide complete solutions to 6 of the 7 questions

1. Let \( f(z) = \begin{cases} \frac{x^{\frac{1}{2}} y^{\frac{3}{2}} + iz^2 y^\frac{1}{2}}{x^2 + y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \)

Show that the Cauchy-Riemann equations hold at \( z = 0 \), but \( f \) is not differentiable at \( z = 0 \).

2. If \( f(z) = u(x, y) + iv(x, y) \) is entire such that \( au + bv \geq c \) for some real numbers \( a, b, \) and \( c \), must \( f \) be constant? Prove your answer.

3. Let \( g \) be a continuous complex-valued function of a real variable on \( [0, 2] \), and for each complex number \( z \) define

\[ F(z) := \int_0^2 e^{z t} g(t) dt. \]

Prove that \( F \) is entire, and find its power series around the origin.

4. Find the Laurent series for the function

\[ f(z) = \frac{z}{(z+1)(z-2)} \]

in each of the following domains:

(a) \( |z| < 1 \)  
(b) \( 1 < |z| < 2 \)  
(c) \( 2 < |z| \)

5. Does there exist a function \( f(z) \) analytic in \( |z| < 1 \) and satisfying

\[ f\left(\frac{1}{2}\right) = \frac{1}{2}, \quad f\left(\frac{1}{3}\right) = \frac{1}{3}, \quad f\left(\frac{1}{4}\right) = \frac{1}{4}, \quad f\left(\frac{1}{5}\right) = \frac{1}{5}, \quad \ldots, \quad f\left(\frac{1}{2n}\right) = \frac{1}{2n}, \quad f\left(\frac{1}{2n+1}\right) = \frac{1}{2n}, \quad \ldots \]

Justify your answer.

6. Show that all roots of \( z^5 - 3z^2 - 1 = 0 \) lie inside the circle \( |z| = 2^{\frac{3}{4}} \) and two of its roots lie inside the circle \( |z| = \frac{3}{4} \).

7. Prove that \( \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 9)^2} dx = \frac{\pi}{6} \).