1. Suppose \( f: K \rightarrow \mathbb{R} \) be a convex function where \( K \subset \mathbb{R}^n \) is a convex set and consider the problem: minimize \( \min_{x \in K} f(x) \).

   (a) Give conditions that guarantee that the problem has a solution.

   (b) Suppose \( x_0 \in K \) is a local solution. Show that \( x_0 \) is also a global solution.

2. Consider the unconstrained optimization problem minimize \( \min_{x \in \mathbb{R}^n} f(x) \) where \( f \) is twice continuously differentiable. Suppose \( x_* \) is a local minimizer of \( f(x) \).

   (a) Prove that for all \( p \in \mathbb{R}^n, p \neq 0 \), \( p^T \nabla f(x_*) \geq 0 \) and hence show that \( \nabla f(x_*) = 0 \).

   (b) Show then that \( p^T \nabla^2 f(x_*) p \geq 0 \) for all \( p \in \mathbb{R}^n, p \neq 0 \).

3. Solve

   minimize \( 2x_1x_2 + x_2^2 - x_1^2 + x_3^2 \)

   subject to \( 4 - x_2^2 - x_3^2 \geq 0 \)

   \( 2 - x_1 - x_3 \geq 0 \)

   \( 2 + x_1 - x_3 \geq 0 \)

   Provide full justification as to why the point you find is a solution.

4. Consider the following.

   **Farkas’ Lemma**

   Let \( A \) be an \( m \times n \) real matrix and \( b \in \mathbb{R}^n \). Then \( b^T y \geq 0 \) for all \( y \) satisfying \( Ay \geq 0 \) if and only if there exists a vector \( u \in \mathbb{R}^m, u \geq 0 \), such that \( A^T u = b \).

   The figure below (based on a similar figure from *Nonlinear Programming: Analysis and Methods*, Mordecai Avriel, Dover Publications, 2003) illustrates Farkas’ Lemma for a \( 3 \times 2 \) matrix \( A \) with rows \( r_1, r_2, \) and \( r_3 \). \( Y \) is the set of vectors that make an acute angle with \( r_1, r_2, \) and \( r_3 \). \( b_1 \) is such that \( b_1^T y \geq 0 \) for all \( y \in Y \). Note \( b_1 \) is a nonnegative combination of \( r_1, r_2, \) and \( r_3, \) but \( b_2 \) is not.
Consider the problem:

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_i(x) \geq 0, \quad 1 \leq i \leq m
\end{align*}
\]

Let \(x_\ast \in \mathbb{R}^n\) and \(I(x_\ast) = \{i : g_i(x_\ast) = 0\}\). Define

\[
F(x_\ast) = \{p : p^T \nabla g_i(x_\ast) \geq 0, \ i \in I(x_\ast)\}
\]

\[D(x_\ast) = \{p : p^T \nabla f(x_\ast) < 0\}.
\]

(a) Explain what it means for \(F(x_\ast) \cap D(x_\ast) = \emptyset\).

(b) Use Farkas’ Lemma to prove \(F(x_\ast) \cap D(x_\ast) = \emptyset\) if and only if there exist \(\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_i \geq 0, \ 1 \leq i \leq m\), such that

\[
\nabla f(x_\ast) = \nabla g(x_\ast)^T \lambda, \ \text{i.e.} \ \nabla f(x_\ast) = \sum_{i=1}^{m} \lambda_i \nabla g_i(x_\ast), \ \text{and} \ \lambda_j g_j(x_\ast) = 0, \ j = 1, 2, \ldots, m.
\]

(Hint: Show \(F(x_\ast) \cap D(x_\ast) = \emptyset\) if and only if for all \(p\) that satisfies \(p^T \nabla g_i(x_\ast) \geq 0\), \(i \in I(x_\ast)\), then \(p^T \nabla f(x_\ast) \geq 0\) and then apply Farkas’ Lemma.

**MTH 638**

1. Let \(G \subset \mathbb{R}^n\) be open and \(f : G \to \mathbb{R}^n\) be a continuous function such that \(D(f)(x)\) exists and is continuous for all \(x \in G\). Suppose there exists a point \(x_\ast \in G\) such that \(f(x_\ast) = 0\).

   (a) Define Newton’s method for the solution of the system \(f(x) = 0\).

   (b) Give conditions on \(f\) and \(D(f)(x)\) under which Newton’s method converges.

2. Let \(f\) have at least 3 continuous derivatives on \([0, 1]\).

   (a) Find the polynomial of degree 2 such that

   \[
   p(0) = f(0), \ p(1) = f(1), \ p'(1) = f'(1).
   \]

   Give the remainder for this interpolant.

   (b) Using the polynomial in part (a) find the quadrature rule for approximating

   \[
   \int_0^1 f(x) \, dx
   \]

   and the error term for this approximation.

   (c) Give the composite quadrature rule for the rule found in part (b) and provide the error term as well. Comment on the usefulness of this composite rule.

3. Let \(f\) be at least twice continuously differentiable on the interval \([a, b]\).

   (a) Explain why for any \(\epsilon > 0\) there is a polynomial \(q(x)\) such that \(\|f' - q\|_\infty < \epsilon\).

   Here \(\|f\|_\infty = \sup \{|f(x)| : a \leq x \leq b\}\).

   (b) Using \(q\) in part (a) show there is a polynomial \(p(x)\) such that \(\|f - p\|_\infty < \epsilon(b - a)\).

4. Consider the scalar differential equation \(y' = f(x, y); \ y(a) = c\). An implicit single step method for approximating a solution to the differential equation is given by

   \[
   u_{i+1} = u_i + h\phi(x_i, u_i, u_{i+1}; h), \quad h > 0, \quad u_0 = c, \quad x_i = a + i \cdot h, \ 0 \leq i \leq n.
   \]
The local truncation error for the method is given by
\[ \tau(x_i, h) = \frac{u_{i+1} - u_i}{h} - \phi(x_i, u_i, u_{i+1}; h). \]

(a) The following implicit single step method

(a1) \[ u_0 = c \]
(a2) \[ u_{i+1} = u_i + \frac{h}{2} \left( f(x_i, u_i) + f(x_{i+1}, u_{i+1}) \right) \]

is known as the trapezoidal method. Show that the local truncation error for this method is \( O(h^2) \), i.e., 2\(^{nd}\) order.

(b) The goal at each step of this method is, given \( u_i \), find \( u_{i+1} \), but equation (a2) is usually nonlinear. Give a method for finding \( u_{i+1} \) and discuss a practical implementation of the trapezoidal method for approximating the solution of a differential equation on an interval \([a, b]\).