Combinatorics Qualifying Examination

May 12, 2008

Name:

The examination consists of two parts. In the first part you have to do five out of the six problems. In the second part you have to do five out of the six problems.

Part I:

1. How many different "words" can we obtain by permuting the letters of "TITTABAWASI"?

2. (a) Find the number of integers between 1 and 10,000 inclusive that are not divisible by 4, 6, or 10.

   (b) At a party seven gentlemen check their hats. In how many ways can their hats be returned so that

   (i) no gentleman receives his own hat?

   (ii) at least one of the gentlemen receives his own hat?

   (iii) at least two of the gentlemen receive their own hats?

3. (i) Use the binomial theorem to prove that

   \[ 3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k. \]

   (ii) Find one binomial coefficient equal to the following expression

   \[ \binom{n}{k} + 3\binom{n}{k-1} + 3\binom{n}{k-2} + \binom{n}{k-3}. \]

   (iii) Prove that for a positive integer \( n \),

   \[ 1 + \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} + ... + \frac{1}{n+1} \binom{n}{n} = \frac{2^{n+1} - 1}{n+1}. \]
4. (i) Let $G$ be a tree on $v \geq 2$ vertices. Prove that $G$ has at least two vertices of degree one.

(ii) Let $G$ be a simple graph on $v$ vertices (i.e. no loops or multiple edges). Call $G$ self-complimentary if $G$ is isomorphic to its complement. Prove that $G$ is self-complimentary if $v \equiv 0, 1 \mod 4$.

5. Let $G = (V, E)$ be a network with source $s$ and sink $t$. Define the notion of a cut $(X, Y)$ in the network $G$. State the relation between the maximal flow in a network and the capacities of all the cuts. Find the value of the maximal flow in the network below.

insert graph

6. Let $X$ and $Y$ be two sets of size $m$ and $n$ respectively. Compute the number of

i) different functions $f : X \to Y$

ii) one to one functions $f : X \to Y$
Part II.

1. Give constructions of the following:

   (i) A symmetric $(7, 3, 1)$-design.
   (ii) A symmetric $(16, 6, 2)$-design.
   (iii) A 2-design with parameters $(49, 7, 1)$.
   (iv) A Hadamard matrix of order 128.
   (v) A symmetric $(63, 31, 15)$-design.

2. (i) State and prove Fisher’s inequality for a $BIBD$.
   (ii) Let $D$ be a symmetric $(v, k, \lambda)$-design with $v$ even. Prove that the order $n = k - \lambda$ must be a square.
   (iii) Let $D$ be a symmetric $(v, k, \lambda)$-design of order $n \geq 2$. Prove that $4n - 1 \leq v \leq n^2 + n + 1$. Are these bounds on $v$ best possible. Justify your answer briefly.

3. (i) Let $H$ be a Hadamard matrix of order $n$. Prove that $\det(H) = n^{n/2}$.
   (ii) Let $H$ be a Hadamard matrix of order $n$. Prove that $n$ must be 1, 2, or a multiple of 4.
   (iii) Prove that the row sum of a regular Hadamard matrix of order $n \geq 4$ is even and not equal to zero. If it is equal to $s$, then $n = s^2$.

4. (i) State the Bruck-Ryser-Chowla Theorem about symmetric designs.
   (ii) Prove that there is no symmetric $(43, 7, 1)$-designs, and no symmetric $(29, 8, 2)$-designs.
(iii) Is the Bruck-Ryser-Chowla condition sufficient for the existence of a symmetric 
(v, k, \lambda)-design? Justify your answer appropriately.

5. Construct three MOLS of order 4, then use these MOLS to construct an affine 
plane of order 4.

6. Let D be a non-trivial symmetric (v, k, \lambda)-design with v > k \geq 2 and let B be 
a block of D. Show that the residual design D^B is a (v - k, v - 1, k, k - \lambda, \lambda)-design 
and that the derived design D_B is a (k, v - 1, k - 1, \lambda, \lambda - 1)-design.