General Instructions:
There are two parts: Part A consists of 4 questions and Part B consists of 5 questions. Begin each question on a separate sheet. Use only one side of the sheets. Clearly indicate Parts A and B.

Part A
Please give complete solutions to all four questions. Provide your NAME on the first sheet.

Question #1
(a) Diseases A and B are prevalent among people in a certain population. It is assumed that 18% of the population will contract disease A sometime during their lifetime, 15% will contract disease B eventually, and 6% will contract both diseases. Find the probability that a randomly chosen person from this population will contract
(i) at least one disease.
(ii) at most one disease.
(iii) exactly one of the diseases.
(b) Four candidates are seeking a vacancy on a school board. If A is twice as likely to be elected as B, B is thrice as likely to be elected as C, and C is twice as likely to be elected as D, what is the probability that B will win?
(c) If A' and B are independent events, prove that A and B are also independent.

Question #2
Let X be a continuous random variable with probability density function
\[ f(x) = \begin{cases} 
5e^{-5x}, & x > 0 \\
0, & \text{elsewhere.}
\end{cases} \]
(a) Find the moment generating function of X.
(b) Find the mean and variance of X.
(c) Find E(e^{3X}).
(d) If Y = 5X - 1, find the moment generating function of Y.
(e) Using your result in (d) or otherwise, find the mean and the variance of Y.

Question #3
Chebyshev’s Theorem: If \( \mu \) and \( \sigma \) are the mean and the standard deviation of a random variable X, then for any positive constant k, \( P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} \). Prove the Chebyshev’s theorem for a continuous random variable X.
Question #4
Suppose $X$ and $Y$ are continuous random variables with joint probability density function

$$f(x, y) = \begin{cases} kxy, & 0 \leq x \leq 1, \ 0 < y < 1, \ x + y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the value of constant $k$.
(b) Find the following probabilities:
   (i) $P(X + Y < \frac{1}{2})$
   (ii) $P(X \leq \frac{1}{2}, Y \leq \frac{1}{2})$
(c) Find the marginal density of $X$.
(d) Find the marginal density of $Y$.
(e) Find the conditional density of $Y$ given $X = x$.
(f) Are $X$ and $Y$ independent? Why or why not?

Part B

Please give complete solutions to all five questions. Provide your NAME on the first sheet.

1. Let $Y_1 \sim b(n_1, p_1)$, $Y_2 \sim b(n_2, p_2)$ be independent. Find a 90 percent confidence interval for $p_1 - p_2$, given $n_1 = n_2 = 100$, $y_1 = 50$, $y_2 = 40$.

2. Suppose $X_1$, $X_2$, ..., $X_n$ is a random sample from each of the distributions having the following probability density functions, find the maximum likelihood estimate of the parameter $\theta$ in each case. Are the estimators consistent?
   (a) $f(x; \theta) = \frac{1}{x} e^{-x-\theta}$, $-\infty < x < \infty$, $-\infty < \theta < \infty$
   (b) $f(x; \theta) = \frac{\theta^x e^{-\theta}}{x!}$, $x = 0, 1, 2, ..., 0 < \theta < \infty$

3. What is the strong law? and what is the weak law? Describe the relationship of “converge in distribution” and “converge in probability”.

4. State the following concepts:
   (a) Sufficient statistics
   (b) Completeness, what property an estimator may have with completeness?
   (c) Minimum sufficient statistics
   (d) Exponential family case
5. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution $N(\mu; \sigma^2)$, with the density

$$f(x, \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

(a) find the maximum likelihood estimate of $\sigma^2$.
(b) find the unbiased minimum variance estimator of $\sigma^2$.
(c) show that $\bar{X}$ and $\sum_{i=1}^{n} (x_i - \bar{X})^2$ are independent.
(d) what is the distribution of $\frac{\bar{X} - \mu}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{X})^2 / [n(n-1)]}}$?
(e) Derive the likelihood ratio test for testing $H_0: \mu = \mu_0$, $H_a: \mu < \mu_0$.
(f) State Neyman-Pearson theorem and determine whether the test derived in part (e) is a uniformly most powerful test? What if the alternative is $H_a: \mu = \mu_0$?