General Instruction: Answer all questions. Questions are not equally weighted. Print your answer, your name, number the pages and number the problems on your work sheets. Necessary statistics tables are provided.

Part A

Question #1
An experiment is to draw a ball from a bag, which contains 5 red balls and 15 blue balls. Find
(a) the probability of getting 3 red balls in 12 drawings with replacement.
(b) the probability of getting the 5th red ball on the 12th drawing with replacement.
(c) the probability of getting 3 red balls in 12 drawings without replacement.

Question #2
Let $X$ and $Y$ be continuous random variable with the following joint density function. Find
\[
f(x,y) = \begin{cases} 
\frac{3}{4} (2 - x - y) & \text{for } 0 < x < 2, 0 < y < 2, \text{and } x + y < 2 \\
0, & \text{otherwise}
\end{cases}
\]
(a) the marginal probability density function of $Y$.
(b) the conditional probability $P(X < 1 \mid Y < 1)$.
(c) the conditional mean and variance of $X$ given $Y = 1$.
(d) the joint p.d.f. of $Z=X+Y$ and $W=X$.

Question #3
Give the names of the following p.d.f.s and describe their relations. Which one of them is related to the Poisson distribution? Describe what variables are modeled by the two distributions in the Poisson process.

(a) $f_2(x,y) = \begin{cases} 
\frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\theta}} & x > 0 \\
0 & \text{otherwise}
\end{cases}$

(b) $f_2(x,y) = \begin{cases} 
\frac{1}{2^{\nu/2} \Gamma(\nu/2)} \frac{x^{\nu/2}}{\nu/2} e^{-\frac{x}{2}} & x > 0 \\
0 & \text{otherwise}
\end{cases}$

(c) $f_3(x,y) = \begin{cases} 
\frac{1}{\theta} e^{-\frac{x}{\theta}} & x > 0 \\
0 & \text{otherwise}
\end{cases}$

Question #4
The number of plans arrive per day at a small airport is a random variable having a Poisson distribution with $\lambda = 28.8$. Calculate
(a) the probability of at least three plans arrive in the next six hours.
(b) the probability that the time between such arrive is at least one hour.
Question #5
For the given bivariate normal distribution below

\[ f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - \rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[ \left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho \left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 \right] \right) \quad \text{where} \quad -\infty < x < \infty \quad \text{and} \quad -\infty < y < \infty \]

(a) show that both of the marginal distributions for X and for Y are normal.
(b) find the covariance of X and Y
(c) can you conclude that if the joint distribution is normal then the marginal distributions are also normal?
(d) if the marginal distributions are normal, does their joint distribution have to be normal? If not explain with an example.
Part B

Question #1
(a) Let $u(X)$ be a non-negative function of the random variable $X$. If $E[u(X)]$ exists, then for every positive constant $c$, prove that $P\left(u(X) \geq c\right) \leq \frac{E[u(X)]}{c}$.

(b) Using your result in (a) or otherwise, prove that $P\left(|X - \mu| \geq k\sigma\right) \leq \frac{1}{k^2}$, $k > 0$ for any random variable $X$ with mean $\mu$ and variance $\sigma^2$.

(c) Let $X$ be a random variable with moment generating function $M(t)$, $0 < t < h$.
   i. Prove that $P(X \geq k) \leq e^{-k}M(t)$.
   ii. Suppose $M(t) = (2t)^{-1}(e^t - e^{-t})$, $0 < t < \infty$. Use the result in (i) to show that $P(X \geq 1) = 0$.
      For this random variable $X$, determine $M(0)$.
   iii. Identify the random variable with the moment generating function $M(t)$ in (ii).

Question #2
(a) Define each of the following concepts for a sequence $\{X_n\}$, $n \geq 1$ of random variables:
   i. Convergence in probability.
   ii. Convergence in distribution.

(b) Suppose the random variable $U_n$ has a gamma distribution with parameters $\alpha = n/2$ and $\beta = 2$. Use Chebyshev’s inequality (or any other method) to prove that the ratio $Y_n = U_n / n^2$ converges in probability to a constant $k$. Find $k$.

(c) If $X_n$ converges in probability to a random variable $X$ and $Y_n$ converges in probability to a random variable $Y$, prove that $X_n + Y_n$ converges in probability to $X + Y$.

(d) Let $Y_1 < Y_2 < \ldots < Y_n$ be the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a distribution with probability density function $f(x) = 4x^3$, $0 < x < 1$ and zero otherwise.
   i. Find $k$ so that the sequence of random variables $W_n = n^4Y_1$ converges in distribution.
   ii. Find the limiting distribution of $W_n$ for the value of $k$ in (i).
   iii. Compute the median of the $X_i$ values. Find the probability that $Y_1$ exceeds the median.

Question #3
Let $X_1, X_2, \ldots, X_n$ denote a random sample from uniform distribution with probability density function $f(x; \theta) = \begin{cases} 1/(4 - 2\theta), & 2\theta+1 < x < 5, \theta > 0 \\ 0, & \text{elsewhere}. \end{cases}$

(a) Find the moment estimator of $\theta$.
(b) Find the maximum likelihood estimator of $\theta$.
(c) Find an unbiased maximum likelihood estimator of $\theta$.
(d) Find an unbiased maximum likelihood estimator of $(4-2\theta)^2$.
(e) When is an estimator $T$ consistent for a parameter $\theta$?
(f) Show that the moment estimator in (a) is consistent.
Question #4
(a) State the Neyman-Pearson theorem.
(b) Suppose a random variable $X$ has the gamma probability density function $f(x; \theta) = \theta e^{-x/\theta}$ for $x > 0$. Consider the simple null hypothesis $H_0 : \theta = \theta_0$ against the alternative hypothesis $H_1 : \theta < \theta_0$. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size $n$ from the distribution. Use the Neyman-Pearson theorem to find the most powerful critical region of size $\alpha$.
(c) Suppose $X_1$ and $X_2$ constitute a random sample of size 2 from the population with probability density function $f(x; \theta) = \theta x^{\theta-1}$, for $0 < x < 1$ and zero otherwise. Suppose the critical region $x_1 x_2 \geq 0.9$ is used to test $H_0 : \theta = 2$ against $H_1 : \theta = 4$.
   i. Find the power of the test in term of the parameter $\theta$.
   ii. What is the significance level of the test? What is the power of the test at $\theta = 4$?
   iii. Comment on the power in (ii) relative to the significance level in (ii).
   iv. Suppose the power in (ii) is too small, suggest two ways to increase the power. Which method, if any, is better and why?

Question #5
Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution $f(x; \theta) = \theta^x e^{-\theta} / x!$, for $x = 0, 1, 2, \ldots$
(a) Show that the likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is based upon the statistic $Y = \sum_{i=1}^{n} X_i$. Show that the rejection region is of the form $Y \leq c_1$ or $Y \geq c_2$.
(b) Obtain the null distribution of $Y = \sum_{i=1}^{n} X_i$.
(c) For $\theta_0 = 0.5$ and $n = 50$, find the approximate significance level of the test that rejects $H_0$ if $Y \leq 15$ or $Y \geq 35$. Use the Central Limit Theorem and ignore continuity correction.