Ph.D. QUALIFYING EXAMINATION
STATISTICS

Time: 9:00-11:00 a.m., 180 minutes
August 8, 2003

General Instructions:
There are two parts: Part A consists of 6 questions and Part B consists of 5 questions. Detailed
solutions (with question number clearly labeled) to each question are to be presented on a separate
sheet. Write on one side only. After finishing, please collate all your work and write down the page
number according to the order of the question numbers.

Part A
1. Let $X$ and $Y$ be discrete random variables with joint probability function

$$f(x, y) = \begin{cases} \frac{1}{30}(x + y) & \text{for } x = 0, 1, 2, 3; y = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

a) Find $P(X \leq 1 | Y \leq 1)$.

b) Find the variance of $2X - Y$.

2. Let $X$ and $Y$ be continuous random variable with joint density function

$$f(x, y) = \begin{cases} \frac{3}{4} (2 - x - y) & \text{for } 0 < x < 2, 0 < y < 2, \text{ and } x + y < 2 \\ 0, & \text{otherwise} \end{cases}$$

a) Find the marginal probability density function of $Y$.

b) Find the conditional probability $P(X < 1 | Y < 1)$.

c) Find the conditional mean and variance of $X$ given $Y = 1$.

3. Let $X$ and $Y$ be continuous random variables with joint density function

$$f(x, y) = \begin{cases} e^{-x-y} & \text{for } 0 < x < \infty, 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

a) Are $X$ and $Y$ independent? Why or Why not?

b) If $Z = X + Y$, find the probability density function of $Z$ and compute $P(Z \leq 6)$.

4. Let $f(x) = 1$, $0 < x < 1$, zero elsewhere, be the probability density function (p.d.f.) of $X$. Find the
distribution function and the p.d.f. of $Y = \sqrt{X}$.

5. If $X$ has a binomial distribution with the parameters $n$ and $\theta$, compute $E(e^{\alpha X})$ (the answer should
be in a closed form).

6. The board of directors of a corporation wishes to purchase “headhunters insurance” to cover the
cost of replacing up to 3 of the corporations high-ranking executives, should they leave during the
next year to take another job. The board wants the insurance policy to pay $1,000,000 \times K^2$, where
$K=0, 1, 2$ or 3 is the number of the three executives that leave within the next year. An
actuary analyzes the past experience of the corporation’s retention of executives at that level, and
estimates the following probabilities for the number who will leave: $P[K=0]=.8$, $P[K=1]=.1$,
$P[K=2]=P[K=3]=.05$. Find the expected payment the insurer will make for the year on this policy.
Part B

7. Suppose $X_1, X_2, ..., X_n$ are a random sample from $N(\mu, \sigma^2)$ and $S^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$,
   
a. Find the limiting distribution of $S^2$ and the limiting distribution of $\frac{n}{n-1} S^2$.
   
b. Find the unique unbiased minimum variance estimator for and $\sigma^2$.
   
c. Show that $\bar{X}$ is the unique unbiased minimum variance estimator of $\mu$.
   
d. Show that $\bar{X}$ and $S^2$ are independent.
   
e. Derive the uniformly most powerful test for $H_0 : \mu = 25$, $H_a : \mu > 25$, with $\alpha = 0.05$ and $n = 10$.
   
f. Find the sample size $n$ such that the power of the test in part e is 0.90 when $H_a : \mu = 23$.

8. Suppose $X_1, X_2, ..., X_n$ are a random sample from the distribution with p.d.f.
   
   $$ f(x; \theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, 2, ..., \quad 0 < \theta < \infty, \text{ zero elsewhere.} $$

   Find the unique unbiased minimum variance estimator of $\theta$. Is the estimator an m.i.e.? And is it
   consistent? Explain.

9. Suppose $X_1, X_2, ..., X_n$ are a random sample from
   
   $$ f(x; \theta) = \frac{1}{2} \theta^3 x^2 e^{-\theta x}, \quad 0 < x < \infty, \text{ zero elsewhere, where } 0 < \theta < \infty. $$

   a. Show that $X_2$ and $\frac{X_2}{\sum_{i=1}^{n} X_i}$ are independent.
   
b. What is the distribution of $\frac{X_2}{\sum_{i=1}^{n} X_i}$?

10. What is the law of large number? Describe the relationship of “converge in distribution” and
    “converge in probability”.

11. Suppose $Y_1 < Y_2 < Y_3 < Y_4 < Y_5$ are an order statistics from $f(x; \theta) = \frac{1}{\theta}, \quad 0 < x < \theta, \text{ zero elsewhere, where } 0 < \theta < \infty$.

   a. Show that both $2Y_3$ and $\varphi(Y_3)$ are unbiased estimator of $\theta$, where $\varphi(y_3) = E(2Y_3 | y_3)$.
   
b. Which estimator is better? Explain.
   
c. Does there exists a unique unbiased minimum estimator of $\theta$? Why?