Numerical Methods for High Dimensional Singular Perturbation Problems

Weiqun Zhang
Wright State University – Lake Campus
weiqun.zhang@wright.edu

Abstract. Improved a’priori bounds and boundary layer tests were extended to two dimensional singular perturbation problems. Boundary layers along x-axis, y-axis, or both are considered. A robust numerical scheme was developed. The numerical error is maintained at the same level for a family of singular perturbation problems with a constant number of mesh points. Numerical experiment supports the theory.

2000 Mathematics Subject Classification: 65N12; 32E25.

We solve numerically the two dimensional SPP

$$\varepsilon (u_{xx} + u_{yy}) + u_x = 0 \quad \text{for } 0 < x, y < 1,$$
$$u(0, y) = y(1 - y), \ u(l, y) = 0,$$
$$u(x,0) = 0 \quad \text{and} \ u(x,1) = 0,$$

where the exact solution is

$$u(x, y) = [y(1-y) - 2\varepsilon x]e^{-\frac{x}{\varepsilon}},$$

Choo and Schultz [2]. According to the boundary layer test of Zhang [5], there is a boundary layer at x=0, which is shown in Figure 1.
The two dimensional SPP (1) is approximated by

\[ u_x = 0, \quad w \varepsilon < x < 1 \quad \text{and} \quad 0 < y < 1, \]
\[ u(1, y) = 0, \]
\[ u(x,0) = 0, \quad u(x,1) = 0, \]  

and

\[ \varepsilon (u_{xx} + u_{yy}) + u_x = 0, \quad 0 < x < w \varepsilon \quad \text{and} \quad 0 < y < 1, \]
\[ u(0, y) = y(1 - y), \quad u(w \varepsilon, y) = 0, \]
\[ u(x,0) = 0, \quad u(x,1) = 0, \]  

Figure 1 Graphs of the solutions of the SPP (2)
We apply the finite difference scheme in Zhang [4] on the following boundary layer adapted mesh along the x direction to solve the 2-D SPP (1).

![Figure 2: Boundary layer adapted 2-D meshes](image)

Choo and Schultz [2] solved the SPP (1) for $\varepsilon$ down to $10^{-3}$ with error level controlled at $10^{-3}$. The new method in this paper produced much better result with error controlled at $10^{-5}$, shown in Table 1.

<table>
<thead>
<tr>
<th>Method</th>
<th>Mesh Points</th>
<th>Max Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform with the central differences</td>
<td>50</td>
<td>1.20*10^{-2}</td>
</tr>
<tr>
<td>Uniform with Choo’s 4th order</td>
<td>50</td>
<td>8.40*10^{-3}</td>
</tr>
<tr>
<td>The new method with the central difference</td>
<td>50</td>
<td>1.23*10^{-5}</td>
</tr>
</tbody>
</table>

Table 1: Comparison among methods with $\varepsilon=10^{-3}$

As it does for a one dimensional SPP, the new method works for very small values of the singular perturbation parameter $\varepsilon$. In Table 2, different numbers of points on the boundary layer in the x direction and a fixed number of points (20) in the y direction were tested. The convergence of the new method was verified.

<table>
<thead>
<tr>
<th>Points on the boundary layer</th>
<th>50</th>
<th>80</th>
<th>110</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max Error when $\varepsilon=10^{-6}$</td>
<td>1.23*10^{-3}</td>
<td>1.19*10^{-3}</td>
<td>9.03*10^{-6}</td>
</tr>
<tr>
<td>Max Error when $\varepsilon=10^{-10}$</td>
<td>1.23*10^{-3}</td>
<td>1.19*10^{-3}</td>
<td>9.03*10^{-6}</td>
</tr>
</tbody>
</table>

Table 2: The convergence of the new method

The number of points in the y direction is 20.
REFERENCES