Analysis Qualifying Exam
August 22, 2019

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

1. Let $m^*(A)$ denote the outer measure of $A \subset \mathbb{R}$. Prove or disprove: If $A \subset B \subset [0,1]$, then $m^*(B - A) = m^*(B) - m^*(A)$.

2. State Egoroff's Theorem, and give an example, with justification, to show that Egoroff's Theorem can fail if the domain has infinite measure.

3. Let $f \geq 0$ be integrable. Consider function F on $\mathbb{R}$ defined by

$$F(x) = \int_{-\infty}^{x} f(y)\, dy.$$

   (a) (3 points) Show that F is continuous.

   (b) (7 points) Is F necessarily Lipschitz? Justify your answer.

4. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{\sqrt{1-x}} & \text{if } x \in [0,1]\backslash\mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$$

   (a) (3 points) Show that f is measurable.

   (b) (4 points) Is f Lebesgue integrable? If yes, find its Lebesgue integral.

   (c) (4 points) Prove or disprove that f is of bounded variation on [0, 1]

5. Show that if f is continuous on $[0,1]$ and f' is bounded on (a,b) everywhere, then f is absolutely continuous.

6. Find all functions $f \in L^3([0,1])$ satisfying the equation

$$\left(\int_0^1 x f(x)\, dx\right)^3 = \frac{4}{25} \int_0^1 f^3(x)\, dx.$$

MTH 636: Provide complete solutions to **only** 5 of the 6 problems.

1. Let $U \subset \mathbb{C}$ be an open disk. A complex-valued function $f = u(x, y) + iv(x, y)$ on U is called *harmonic* if

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 0.$$

   (a) Prove that any holomorphic function is harmonic.

   (b) Prove that if f is a real-valued harmonic function on U, then there exists a holomorphic function g on U such that f is the real part of g.

2. State and prove the Fundamental Theorem of Algebra.

3. Let $U = \{z \in \mathbb{C} : |z| < 3, \operatorname{Im}(z) > 0\} \subset \mathbb{C}$, and let f be a holomorphic nowhere vanishing function on U. Show that there exists a holomorphic function g on U such that

$$f(z) = \frac{1}{g(z)^2}$$

   for all $z \in U$.

4. Let $r > 0$, let $z_0 \in \mathbb{C}$, let $D'(z_0, r)$ be the punctured disk of radius r around z_0, and let f be a function holomorphic on $D'(z_0, r)$. Suppose that there exists a positive integer N and a real number α such that $\alpha < N + 1$ and

$$|f(z - z_0)| < c|z - z_0|^{-\alpha}$$

   for all $z \in D'(z_0, r)$, where c is some real constant. Show that f either has a removable singularity at z_0, or a pole of order no larger than N.

5. Let $a > 0$. Use the Residue Theorem to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(a^2 + x^2)^2}.$$

6. Find a fractional linear transformation f such that $f(0) = \infty$, $f(i) = 1$, and such that f maps the unit circle around the origin to itself.