## Analysis Qualifying Exam

**August 26, 9:00-12:00, 1995**

**Section A** Answer 10 of the 12.

1. Prove that a sequence $\langle \vec{x}_m \rangle$ in $\mathbb{R}^n$ converges to $\vec{x}$ in $\mathbb{R}^n$ if and only if $x_{mj} \to x_j$ for each $j = 1, 2, \ldots, n$ where $\vec{x}_m = (x_{m1}, x_{m2}, \ldots, x_{mn})$ and $\vec{x} = (x_1, x_2, \ldots, x_n)$.

2. Prove the continuity of the following function. State carefully the theorem(s) used.

$$f(x) = \sum_{n=0}^{\infty} \frac{\sin nx}{1+n^2}, \quad \text{for } x \text{ real.}$$

3. Is

$$f(x,y) = \begin{cases} \dfrac{x(x^2-y^2)}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

differentiable? Prove your answer.

4. (a) The Mean Value Theorem for Integrals for functions of one variable states that if $f$ is continuous on $[a,b]$, then there exists a point $x_0 \in [a,b]$ such that $\displaystyle\int_a^b f(x)\,dx = f(x_0)(b-a)$. Is there a parallel theorem for functions defined on a compact convex subset of $\mathbb{R}^n$? If your answer is yes state and prove the result. If no, give an example to explain why such a result can not be valid.

   (b) What conditions on the partial derivatives at a point imply that the point is a saddle point?

5. For each $n$ let $f_n(x) = x^n$, for $x \in [0,1]$. Prove that $f_n(x) \to f(x)$ for all $x \in [0,1]$ where $f(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < 1 \\ 1 & \text{if} \quad x = 1. \end{cases}$ Is the convergence uniform? Prove your answer. Does $\displaystyle\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx$? Does this follow from any of the theorems you have learned? If yes, state the theorem. Or does this contradict any theorems?

6. Let $E$ be a countable subset of $\mathbb{R}$.

   (a) Using the definition of outer measure prove that $m^*E = 0$.

   (b) Is $E$ measurable and what is its measure? Prove your answer.

7. Let $f$ be an integrable function on $\mathbb{R}$.

   (a) Prove that there exists a set $E_\infty$ of measure zero such that $f$ is finite for all $x \notin E_\infty$.

   (b) Prove that there exists a sequence $\langle f_n \rangle$ of bounded measurable functions, each vanishing outside a set of finite measure such that $f_n \to f$ a.e.

8. If $F$ is absolutely continuous on $[0,1]$ and $F'(x) = \dfrac{1}{\sqrt{x}}$ a.e., how would you compute $F(x)$ for $x \in [0,1]$? Justify your arguments.

9. Give an example, if possible, of each of the following. Prove your answer.
   (a) A function $f$ on $[0, \infty)$ such that $\int_0^\infty f(x)\,dx$ exists as an improper Riemann integral but $f$ is not Lebesgue integrable on $[0, \infty)$.
   (b) A Lebesgue integrable function that is not Riemann integrable.
   (c) A sequence $\{f_n\}$ of nonnegative integrable functions defined on $[0, 1]$ such that $\int_{[0,1]} f_n \to 0$, but the sequence does not converge.

10. Find the number of solutions of $f(z) = (z^6 + 2z - 1) - 5z^3 = 0$ interior to the unit circle.

11. (a) State Fatou's Lemma.
    (b) State the Dominated Convergence Theorem, and use Fatou's Lemma to prove it.

12. Let $f(z)$ be an entire function such that $|f(z)| \leq 1 + |z|^{1996}$ for all complex numbers $z$. Prove that $f(z)$ is a polynomial.

**Section B** Select one of the two parts

**Part 1** Answer 2 of the three

1. Let $\mu$ denote Lebesgue measure on $[0, 1]$ and $f$ a nonnegative integrable function on $[0, 1]$.
   (a) Define the set function $\nu$ by $\nu(E) = \int_E f\,d\mu$ where $E$ is a measurable subset of $[0, 1]$. Show $\nu$ is a measure on $[0, 1]$.
   (b) Show $\displaystyle\int_E g\,d\nu = \int_E gf\,d\mu$.

2. Let $(X, M, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be integrable. Prove Chebyshev's inequality: for every $\lambda > 0$

$$\mu(\{x \in X : |f(x)| > \lambda\}) \leq \lambda^{-1} \int_X |f|\,d\mu.$$

3. (a) Let $f$ be a sunction such that all the indicated norms are defined. What are $\|f\|_p$, for $1 \leq p \leq \infty$.
   (b) Discuss the containments amont the $L^p$, $(1 \leq p \leq \infty)$ spaces over a finite measure space. Justify your answer.
   (c) Do part (b) for an infinite measure space.

**Part 2** Answer 2 of 3

1. Find a conformal mapping of the upper half plane, $\operatorname{Im} z > 0$, onto the disk $|w| < R$ such that a given point $\alpha$ in the upper half plane is mapped to the center of the disk. Is your answer unique? Explain.

2. Find the function $u(x, y)$ that is harmonic in the unit disk $|z| < 1$ and takes on the boundary values $u(\theta) = u(\cos\theta, \sin\theta) = \theta/2$ for $-\pi < \theta < \pi$.

3. Prove: if the radius of convergence of the series $f(z) = \sum a_n z^n$ is $R$, then $f(z)$ has at least one singular point on the circle $|z| = R$.