Analysis Qualifying Exam August 19, 1997, 5:00-9:00

Section A Answer 10 of the 12.

1. Consider the sequence of functions

$$f_n(x) = (n+1)x(1-x)^n; \quad 0 \le x \le 1.$$

   (a) Determine, with proof,the pointwise limit of $\{f_n(x)\}$ and where if at all the convergence is uniform.

   (b) Explain, without using uniform convergence, why

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$$

   where $f(x)$ is the pointwise limit of $\{f_n(x)\}$.

2. We say a set $X \subset \mathbb{R}^n$ is sequentially compact if every sequence in X has a convergent subsequence which converges to a limit in X. Let $f : X \to \mathbb{R}^m$. Show that if f is continuous on X, then f is uniformly continuous.

3. Let $\vec{u} \in \mathbb{R}^n$ be such that $\| \vec{u} \| = 1$, (Euclidean norm), and $f : V \to \mathbb{R}$, V an open subset of $\mathbb{R}^n$. The directional derivative of f at $\vec{a} \in V$ in the direction $\vec{u}$ is defined by

$$D_{\vec{u}} f(\vec{a}) = \lim_{t\to 0} \frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t}$$

   when this limit exists. Prove or find a counterexample: If $D_{\vec{u}} f(\vec{a})$ exists for all directions $\vec{u}$, then f is differentiable and hence continuous at $\vec{a}$.

4. Suppose $f : [0,1] \times [0,1] \to \mathbb{R}$ is such that

$$\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx$$

   exists. Does this necessarily imply that f is integrable on $[0,1] \times [0,1]$? Prove or find a counterexample.

5. Give the development of Lebesgue measure and the Lebesgue integral starting with the definition of outer measure and concluding with the important convergence theorems. State carefully the pertinent definitions and theorems.

6. (a) Is the function $f(t) = t^{-2/3}$ integrable in the Riemann sense over the interval $[0,1]$. Give a reason for your answer.

   (b) Is the function $f(t) = t^{-2/3}$ Lebesgue integrable on the interval $[0,1]$? Prove your answer, indicating any convergence theorem you are using.

   (c) Is the function $g(t) = t^{1/3}$ absolutely continuous on $[0,1]$? Justify your answer.

7. (a) Suppose f is integrable on $[a,b]$. Show that the function $F(x) = \int_a^x f(t)\,dt$ is a continuous function of bounded variation on $[a,b]$.

   (b) What do you know about the differentiability of F? Explain.

(c) Suppose f is bounded and measurable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt + F(a)$. For

$$f_n(x) = \frac{F(x + 1/n) - F(x)}{1/n}$$

show

$$\int_a^c F'(x)\,dx = \lim_{n\to\infty} \int_a^c f_n(x)\,dx = \int_a^c f(x)\,dx$$

for all $c \in [a, b]$.

8. (a) Describe the space denoted by $L^p[0,1]$. Indicate it definition and basic properties.

   (b) Suppose F is a bounded linear functional on L^p. What does that mean?

   (c) Given a bounded measurable function on $[0,1]$, is it true or false that there is a bounded sequence of step functions which converge a.e. to f. Prove your answer.

   (d) Suppose that there is an integrable function g on $[0,1]$ such that $F(\psi) = \int_0^1 g\psi$ for each step function ψ on $[0,1]$. Prove that

   $$F(f) = \int_0^1 gf$$

   for each bounded measurable function f on $[0,1]$.

9. Find all entire functions f which have a zero of order two at $x = 0$, satisfy the condition $|f'(z)| \leq 6|z|$ for all z, and are such that $f(i) = -2$.

10. Show that if the analytic function $w = f(z)$ maps a domain $\mathcal{D}$ on the $C = \{z : z = t + it^2, 0 \leq t \leq 1\}$ then f must be constant through out $\mathcal{D}$.

11. Find the value of the following integrals.

    (a) $\int_C \frac{\cos z}{3z - \pi}\,dx$, where C is the circle $|z| = 1$ with positive orientation.

    (b) $\int_C \frac{e^{z^2}}{(z - i)^3}\,dz$, where C is the circle $|z| = 2$ with positive orientation.

12. Suppose $f(z)$ is analytic on $\mathcal{D} = \{z : |z| \leq 1\}$ and satisfies $|f(z)| < 1$ for $|z| < 1$.

    (a) Prove that the equation $f(z) = z$ has exactly one root z_0 in $\{z : |z| < 1\}$.

    (b) Fine the value of $\frac{1}{2\pi i} \oint_{|z|=1} \frac{f'(z)}{f(z) - z_0}\,dz$, where z_0 is given in part (a).

Section B Select one of the two parts

Part 1 Answer 2 of the three

1. Let f be an integrable function on the measure space $(X, \mathcal{A}, \mu)$.

   (a) Show that the set $\{x : f(x) \neq 0\}$ is of σ-finite measure.

   (b) There exists a sequence $\{\phi_n\}$ of simple functions, each of which vanishes outside a set of finite measure such that
   $$\int |f - \phi_n|\, d\mu \to 0$$
   as $n \to \infty$.

2. Determine whether each of the following statement is true. Prove your answer.

   (a) Convergence in measure implies convergence almost everywhere.

   (b) Convergence in almost everywhere implies convergence in measure.

3. (a) Show by examples that a set of measure 0 in a signed measure space may not be a null set, and that a set of positive measure may not be a positive set.

   (b) Give an example of measure space $(X, \mathcal{A}, \mu)$ and a measure ν defined on $\mathcal{A}$, absolutely continuous with respect to μ, but no measurable function f such that $\nu(E) = \int_E f\, d\mu$ for all $E \in \mathcal{A}$.

Part 2 Answer 2 of the three

1. Suppose $u_1(z)$ and $u_2(z)$ are harmonic in a simply connected domain $\mathcal{D}$ with $u_1(z)u_2(z) \equiv 0$ in $\mathcal{D}$. Prove that either $u_1(z) \equiv 0$ or $u_2(z) \equiv 0$ in $\mathcal{D}$.

2. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the disks $|z| < 1$ and $|z - 2| < 1$, respectively, and let $f_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ and $f_2(z) = i\pi + \sum_{n=1}^{\infty} (-1)^n \frac{(z-2)^n}{n}$. Show that $(f_1, \mathcal{D}_1)$ and $(f_2, \mathcal{D}_2)$ are analytic continuations of each other.

3. Show that $\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)$ is entire if and only if $\sum_{n=1}^{\infty} \frac{1}{z - a_n}$ is meromorphic.