ANALYSIS QUALIFYING EXAMINATION
AUGUST 18, 2000
1-5 P.M.

Section A: Do both parts.

Part 1 Do (only) *five* out of the six problems.

1. (a) Rephrase the statement "*a set of positive measure almost contains a closed interval of positive length*" in a mathematically precise way, and then prove it.
   (b) Prove that a set E of finite outer measure ($m^*E < \infty$) is measurable iff for every $\epsilon > 0$ there exist a finite collection of open intervals $I_1, I_2, \ldots, I_n$ such that

$$m^*\left(E\triangle\left(\bigcup_{j=1}^{n} I_n\right)\right) < \epsilon.$$

2. (a) Show that a real-valued function f on $\mathbb{R}$ is measurable iff $f^{-1}(B)$ is measurable for all Borel sets $B \subseteq \mathbb{R}$.
   (b) Show that a simple function f is measurable iff $\{x : f(x) = a\}$ is measurable for all $a \in \mathbb{R}$. Is this still true for an arbitrary function? Prove your answer.

3. (a) Give an example of a function (class) $f \in L^2[1,\infty) \setminus L^1[1,\infty)$, and a $g \in L^1[1,\infty) \setminus L^2[1,\infty)$. Justify your answer
   (b) Do the same for $L^q[1,\infty)$ and $L^p[1,\infty)$, for $1 \leq p < q < \infty$. Prove your answer.

4. (a) Prove that a function with bounded derivative on a closed interval is of bounded variation.
   (b) Give an example of a continuous function on $[0,1]$ that is not of bounded variation.

5. (a) Let $\{f_n\}$ be a sequence in $L^1[0,1]$ such that $f_n \to f$ in $L^1[0,1]$. Prove that $f_n \to f$ in measure.
   (b) Is the converse true? Prove your answer.

6. (a) Give an example of an absolutely continuous and one that is not absolutely continuous but is of bounded variation on $[0,1]$.
   (b) Let f be a function of bounded variation on $[a,b]$ $a, b \in \mathbb{R}$, $a < b$. Show that there is an absolutely continuous function g and a function h such that $h' = 0$ a.e., and $f = g + h$.

Part 2 Do (only) *five* out of the six problems.

1. (a) If $|f(z)|$ is constant in a domain $\mathcal{D}$ and f is analytic in $\mathcal{D}$, show that $f(z)$ is constant in $\mathcal{D}$.
   (b) Find a branch of $\log(z^2+1)$ that is analytic at $z = 0$ and takes the value of $2\pi i$ there.

2. Find the value of $\int_C \frac{e^z}{z^2(z^2-9)}\,dz$ where $C = \{z\colon |z| = 1\}$.

3. (a) State and prove Liouville's Theorem
   (b) Let f be entire and suppose that $\operatorname{Re} f(z) \le \operatorname{Im} f(z)$ for all $z \in \mathbb{C}$. Prove that f is constant.

4. (a) What is a singularity of a complex valued function (defined on an open connected subset of the complex plane)?
   (b) How many kinds of singularities are there? State the definition and give an example of each kind.

5. Evaluate the integral $\int_0^\infty \frac{x^2+1}{x^4+1}\,dx$.

6. (a) State Rouche's Theorem
   (b) Let $p(z) = z^5 + 6z^3 + 2z + 10$. Show that all five zeros of $p(z)$ lie in the set $\{z\colon 1 < |z| < 3\}$.

Section B: Select *one* of the two parts.

Part 1 Do (only) *two* out of the three problems.

1. Let $\mu << m$, where m is the Lebesque measure on $\mathbb{R}$. Define $F(x) = \mu(-\infty, x]$. Show that F is absolutely continuous.

2. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be finite measure spaces. Discuss the development of a product measure $\mu_1 \times \mu_2$ on $\Sigma_1 \times \Sigma_2$ and state Fubini's Theorem.

3. Let (X, M) be a measure space, and let μ, ν, be two positive measures on M such that $\mu(X) + \nu(X) < \infty$. Assume $\mu << \nu$ and let g be the Radon-Nikodym derivative of μ with respect to ν. Show that $L^p(\nu) \subset L^1(\mu)$ if and only if $g \in L^q(\nu)$, $\frac{1}{p} + \frac{1}{q} = 1$.

Part 2 Do (only) *two* out of the three problems.

1. Suppose $\langle u_n(z)\rangle$ is a sequence of harmonic functions that converges uniformly on all compact subsets of a domain $\mathcal{D}$ to a function $u(z)$. Then prove that $u(z)$ is harmonic throughout $\mathcal{D}$.

2. Given a set of real numbers $0 \le \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$, construct a function $f(z)$ satisfying (i) f(z) is analytic in $|z| < 1$ and (ii) the only singular points of $f(z)$ on the unit circle are at $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}$. Prove these assertions for your choice of $f(z)$.

3. Prove the Weierstrass' Theorem: Given any complex sequence having no finite limit point, there exists an entire function that has zeros at these points and only these points.