

## Analysis Qualifying Exam: August, 2010

**MTH 632:** Provide complete solutions to 5 of the 6 questions.

Notation:  $\mathbb{Q}$  denotes the set of rational numbers and  $\mathbb{R}$  denotes the set of real numbers.

1. Let  $E \subset \mathbb{R}$  be such that  $m(E) < \infty$ . Suppose  $f$  and the sequence  $\langle f_n \rangle$  be measurable functions on  $E$  such that  $f_n \rightarrow f$  a.e. Show there exists measurable sets  $G, E_k \subset E$  such that  $E = G \cup (\cup_{k=1}^{\infty} E_k)$ ,  $m(G) = 0$ , and  $f_n \rightarrow f$  uniformly on each  $E_k$ .
2. Let  $\mathcal{M}$  be the  $\sigma$ -algebra of Lebesgue measurable sets in  $\mathbb{R}$  and  $f$  be a function defined on  $\mathbb{R}$  with values in the set of extended real numbers.
  - (a) Show that if  $f$  is measurable then  $f^{-1}(\{\infty\}) \in \mathcal{M}$  and  $f^{-1}(O) \in \mathcal{M}$  for all open subsets  $O \subset \mathbb{R}$ .
  - (b) Show that if  $f^{-1}(\{\infty\}) \in \mathcal{M}$  and  $f^{-1}(B) \in \mathcal{M}$  for all Borel sets  $B \subseteq \mathbb{R}$ , then  $f$  is measurable.

3. Let  $f \in L^1[0, \infty)$  and define

$$g(y) = \int_0^{\infty} f(x) \sin(xy) dx.$$

Show that  $g$  is a bounded and continuous function of  $y$  on all of  $\mathbb{R}$ .

4. Let  $f$  be a real-valued integrable function defined on  $\mathbb{R}$ . Let  $\langle E_n \rangle$  be a sequence of measurable sets such that  $\lim_{n \rightarrow \infty} m(E_n) = 0$ . Show that  $\lim_{n \rightarrow \infty} \int_{E_n} f dm = 0$ .
5. Suppose  $f$  and  $g$  are absolutely continuous functions defined of the interval  $[a, b]$ . Show  $fg$ , their product, is also absolutely continuous.
6. Let  $E \subset \mathbb{R}$  have finite Lebesgue measure.
  - (a) Let  $1 \leq p < q \leq \infty$ . Show  $L^q(E) \subset L^p(E)$  and  $\|f\|_p \leq \|f\|_q (m(E))^{1/p-1/q}$  for any  $f \in L^p(E)$ .
  - (b) If  $1 \leq p < q < r \leq \infty$ . Show  $L^q(E) \subset L^p(E) + L^r(E)$ . Explain why this show that each  $f \in L^q(E)$  can be written as a sum of two functions,  $f = g + h$ ,  $g \in L^p(E)$ ,  $h \in L^r(E)$ . (Hint: For  $f \in L^q(E)$ , consider the two sets  $G = \{x \in E : |f(x)| > 1\}$  and  $H = \{x \in E : |f(x)| \leq 1\}$ .)