Analysis Qualifying Examination August 23, 2012

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

1. Show that an extended real-valued function f on a measurable set E is measurable if and only if $f^{-1}\{\infty\}$ and $f^{-1}\{-\infty\}$ are measurable and for each Borel set A, $f^{-1}(A)$ is measurable.

2. Let $\{f_n\}$ be a sequence of intergrable functions on E for which $f_n \to f$ a.e. on E and f is integrable over E. Show that $\int_E |f_n - f| \to 0$ if and only if $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$.

3. Let f be intergrable over $(-\infty, \infty)$. Show that $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\cos(nx)\, dx = 0$.

4. Prove each of the following for a function f integrable over a measurable set E.

   (i) If $\{E_n\}_{n=1}^{\infty}$ is an ascending countable collection of measurable subsets of E, then $\int_{\bigcup_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f$.

   (ii) If $\{E_n\}_{n=1}^{\infty}$ is a descending countable collection of measurable subsets of E, then $\int_{\bigcap_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f$.

5. A monotone function h on $[a, b]$ is called singular if $h' = 0$ a.e. Show that any monotone increasing function f on $[a, b]$ is the sum of an absolutely continuous function g and a singular function h.

6. Let $E \subset \mathbb{R}$ have finite Lebesgue measure and $1 \leq p < q \leq \infty$. Show that $L^q(E) \subset L^p(E)$ and $\| f \|_p \leq \| f \|_q \, (m(E))^{\frac{1}{p}-\frac{1}{q}}$ for any $f \in L^p(E)$.

MTH 636: Provide complete solutions to **only** 7 of the 8 problems.

1. Show that for any function $f = u + iv$ analytic on $\mathbb{C}$ we have $D \circ \overline{D} f(z) = \triangle u + i \triangle v = 0$ where $D = \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$ and $\triangle$ is the Laplacian operator.

2. Find the set of points z at which $f(z) = |z^2|$ is differentiable. Justify your answer.

3. Show that $2e, e = 2.718...$ is an upper bound for $|\int_C e^z dz|$ where C is the straight line contour joining the points $1 + i$ and $1 - i$.

4. If f is analytic for $|z| \leq 1$ then show that

$$\frac{1}{\pi} \int_{|z| \leq 1} f(x + iy) dx dy = f(0).$$

5. (a) If $f(z) = g(z)/h(z)$ where g and h are analytic at $z = \alpha$ but h has a simple zero at $z = \alpha$, then show that the residue of f at α is given by

$$a_{-1} = \frac{g(\alpha)}{h'(\alpha)}.$$

   (b) Verify the following equation using the result of 5(a)

$$\int_0^{2\pi} \frac{d\theta}{3 + \cos\theta} = \frac{\pi}{\sqrt{2}}.$$

6. Find the series representation of

$$f(z) = \frac{2}{z^2 - 4z + 3}$$

in the following three annuli centered at 0: (i) $0 < |z| < 1$, (ii) $1 < |z| < 3$, (iii) $3 < |z|$.

7. Use Rouche's Theorem to show that all zeros of the complex-valued polynomial

$$p(z) = 6z^7 - 2z^4 + 3$$

are in the interior of the unit disk.

8. Let $a > 0$. Show that

$$\int_{-\infty}^{+\infty} \frac{\cos x}{a^2 - x^2} dx = \pi \frac{\sin a}{a},$$

where the integral converges in the sense of Cauchy.