Analysis Qualifying Exam January 13, 1996, 8:00-12:00

Section A Answer 10 of the 12.

1. (a) State what it means for $f : [a,b] \to \mathbb{R}$ to be uniformly continuous on $[a,b]$.
   (b) State what it means for $f : [a,b] \to \mathbb{R}$ to be Riemann integrable on $[a,b]$.
   (c) Show if f is uniformly continuous on $[a,b]$, then it is Riemann integrable.

2. Let $f_n(x) = \frac{x}{n}e^{-x/n}$ for $x \geq 0$.
   (a) Prove $f_n \to 0$ pointwise for $x \geq 0$.
   (b) Is the convergence in (a) uniform? Prove your answer.
   (c) Where does $f_n \to 0$ uniformly? Prove your answer.

3. Let $f(x,y) = (xy - x + 1, x^2y + 2y - 2)$ for all $(x,y) \in \mathbb{R}^2$.
   (a) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that
   $$f(1+h, 1+k) - f(1,1) = T(h,k) + \epsilon(h,k),$$
   where $\epsilon(h,k)/\|(h,k)\| \to (0,0)$ ans $(h,k) \to (0,0)$.
   (b) Find a linear transformation S that approximates f^{-1} the way T approximates f in part (a).

4. By integrating the function $f(z) = z^{-1}$ over a suitably parameterized ellipse γ and a suitable circle $\bar{\gamma}$, prove that
   $$\int_0^{2\pi} \frac{1}{a^2\cos^2 t + b^2 \sin^2 t}\,dt = \frac{2\pi}{ab} \quad (a > 0,\ b > 0).$$

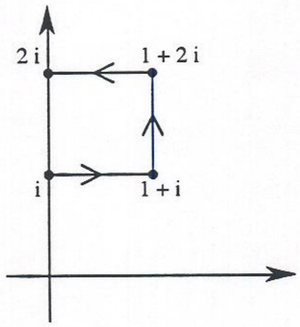
5. Suppose that f is a real-valued function defined on $[0,1]$, satisfying $|f(x_1) - f(x_2)| \leq M|x_1 - x_2|^\alpha$ for all $x_1, x_2 \in [0,1]$. Here $M > 0$ and $\alpha > 0$.
   (a) Show if $0 < \alpha \leq 1$ then f is absolutely continuous.
   (b) Show if $\alpha > 1$ then f is constant.

6. Let AC denote the space of absolutely continuous functions on $[0,1]$, and BV denote the space of functions of bounded variation on $[0,1]$. Does one space contain the other? Prove your answer. If containment exists, is it proper? Prove your answer.

7. Let f be absolutely continuous on $[a,b]$ and $E \subset [a,b]$ be measurable. Show $f(E)$ is measurable.

8. Let $f : [0,1] \to \mathbb{R}$ be continous and let $M = \sup_{x\in[0,1]} |f(x)|$. Show that
   $$M = \lim_{p\to\infty} \left(\int_0^1 |f(x)|^p\,dx \right)^{1/p}.$$

9. (a) Use the definition of a complex integral to evaluate $\int_C z^2\,dz$ if C is the curve from i to $2i$ on the right side.



   (b) Is the integral in part (a) independent of the path? If so calculate it in another way than you did in part (a). Justify your answers.

10. Prove that if $f(z)$ is an entire function and $M(\rho) = \max\{|f(z)| : |z| = \rho\}$, is such that $M(\rho) \leq L\,\rho^k$, then $f(z)$ is a polynomial of degree at most k.

11. Let f be entire and suppose that $\mathrm{Re} f(z) \leq \mathrm{Im} f(z)$ for all $z \in \mathbb{C}$. Prove that f is constant.

12. (a) State Rouche's Theorem.

   (b) Let $p(z) = z^5 + 6z^3 + 2z + 10$. Show that all five zeros of $p(z)$ lie in the set $\{z : 1 < |z| < 3\}$.

Section B Select one of the two parts

Part 1 Answer 2 of the 3

1. Let $\mu << m$, where m is Lebesgue measure on $\mathbb{R}$. Define $F(x) = \mu(-\infty, x]$. Show that F is absolutely continuous.

2. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be finite measure spaces. Discuss the development of a product measure $\mu_1 \times \mu_2$ on $\Sigma_1 \times \Sigma_2$ and state Fubini's Theorem.

3. Let $(X, \mathcal{M})$ be a measure space, and let μ, ν be two positive measures on $\mathcal{M}$ such that $\mu(X) + \nu(X) < \infty$. Assume $\mu << \nu$ and let g be the Radon-Nikodym derivative of μ with respect to ν. Show that $\mathrm{L}^p(\nu) \subset \mathrm{L}^1(\mu)$ if and only if $g \in \mathrm{L}^q(\nu)$, $\frac{1}{p} + \frac{1}{q} = 1$.

Part 2 Answer 2 of the 3

1. Suppose $\langle u_n(z) \rangle$ is a sequence of harmonic functions that converges uniformly on all compact subsets of a domain $\mathcal{D}$ to a function $u(z)$. Then prove that $u(z)$ is harmonic throughout $\mathcal{D}$.

2. Given a set of real numbers $0 \leq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$ construct a function $f(z)$ satisfying (i) $f(z)$ is analytic in $|z| < 1$ and (ii) the only singular points of $f(z)$ on the unit circle are at $e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}$. Prove these assertions for your choice of $f(z)$.

3. Prove the Weierstrass' Theorem: Given any complex sequence having no finite limit point, there exists an entire function that has zeros at these points and only these points.