Analysis Qualifying Exam
January 4, 2011, 13:00 - 17:00

Name: ______________________________

Part 1 Do only *five* of the six problems.

(1) Suppose $A \subseteq \mathbb{R}$ and there are Borél sets $B_1,\ B_2$ such that $m(B_1) = m(B_2)$ and $B_1 \subseteq A \subseteq B_2$. Answer the following questions with proofs.
  (a) Must A be measurable?
  (b) If the answer to part (a) is negative, is it possible to impose some condition(s) so that A is measurable.

(2) (a) For an extended real-valued function f, what do we mean by f being Borél measurable?
  (b) Must a continuous real-valued function on $\mathbb{R}$ be Borél measurable? Prove your answer.
  (c) Must a nondecreasing function defined on $\mathbb{R}$ be Borél measurable? Prove your answer.

(3) (a) Let $E \subseteq \mathbb{R}$ be a measurable set with *finite* measure and $f : E \to \mathbb{R}$ a measurable function. Show that for each $\epsilon > 0$, there is an M such that $m(\{x \in E : |f(x)| > M\}) < \epsilon$. (*Hint: Use f to break up E as a countable union.*)
  (b) Let $\{f_n\}$ and $\{g_n\}$ be sequences of measurable functions on a measurable set E with $m(E) < \infty$. Suppose $f_n \to f$ in measure and $g_n \to g$ in measure. User part (a) to prove that $f_n g_n \to fg$ in measure.

(4) Let $f \in L^2(\mathbb{R})$ (i.e., f is measurable on $\mathbb{R}$ and $\int_{\mathbb{R}} |f(t)|^2\ dt < \infty$). Answer the following questions with justification.
  (a) Is it true that $f \in L^1(\mathbb{R})$?
  (b) For each $x \in \mathbb{R}$, does $\int_{x-\delta}^{x+\delta} f(t)\ dt$ make sense for all $\delta > 0$?
  (c) How big is the set of all $x \in \mathbb{R}$ such that $\lim_{\delta \to 0^+} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(t)\ dt$ exists?
  (d) Find $\lim_{\delta \to 0^+} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(t)\ dt$ wherever exists.

(5) (a) If exists, give an example of a Lebesgue integrable function $f \in (L^1[0,1]) \setminus (L^\infty[0,1])$. Prove your answer, or show that no such example exists.
  (b) Let $p < q$ be positive numbers ≥ 1 be fixed. Are $L^p[0,1]$ and $L^q[0,1]$ comparable? Prove your answers or give counterexamples as needed.
  (c) What about $L^p[0,\infty)$ and $L^q[0,\infty)$?

(6) Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ \frac{1}{\sqrt{x}} & \text{if } x \text{ is irrational.} \end{cases}$
Answer each of the following with proofs.
  (a) Is $f \in L^p[0,\infty]$ for any $p \geq 1$?
  (b) Find all $p \in [1,\infty]$ such that f defines a bounded linear functional on $L^p[0,1]$.

(Please turn over for part 2.)

Part 2 Do only <u>*five*</u> of the six problems.

(1) (a) State the Cauchy-Riemann equations for a complex function $f(z) = u(x, y) + iv(x, y)$.
(b) Let $f(z) = u(x, y)+iv(x, y)$ be defined in some open set G containing the point z_0. Suppose the first partial derivatives of u and v exist on G, are continuous at z_0, and satisfy the Cauchy-Riemann equations at z_0. Is it true that f is differentiable at z_0? Justify your answer.

(2) Suppose that f is analytic at each point of the closed disk $|z| \leq 1$ and that $f(0) = 0$. Prove that the function

$$F(z) = \begin{cases} f(z)/z, & z \neq 0, \\ f'(0), & z = 0, \end{cases}$$

is analytic on $|z| \leq 1$. (Hint: Note that

$$G(z) := \oint_{|\zeta|=1} \frac{f(\zeta)/\zeta}{\zeta - z}\, d\zeta$$

is analytic in the open disk $|z| < 1$.)

(3) Does there exist an entire function such that

$$f(z) = \begin{cases} \frac{1}{z} & \text{if} |z| > 2 \\ e^z & \text{if} |z| < 1? \end{cases}$$

Justify your answer.

(4) Prove that all the roots of $z^4 - z^3 + 7 = 0$ lie in the annulus $1 < |z| < 2$.

(5) Compute

$$\oint_{|z|=2} \frac{1}{z^4 - z^3 + 7}\, dz.$$

Justify your answer.

(6) Compute

$$\int_0^\infty \frac{\sin(x^3)}{x}\, dx.$$

Justify your answer.