

## Analysis Qualifying Examination January 10, 2014

**MTH 632:** Provide complete solutions to **only** 5 of the 6 problems.

1. Prove that if  $E \subset [a, b]$  and  $m^*(E) = 0$  then the complement of  $E$  in  $[a, b]$  is dense in  $[a, b]$ .
2. Let  $\{f_n\}$  be a sequence of real-valued measurable functions defined on  $[0, 1]$ . Show that there exists a sequence of positive real numbers  $\{a_n\}$  such that  $a_n f_n \rightarrow 0$  a.e.  
Hint: Use Lusin's Theorem.
3. Let  $f$  be non-negative, bounded and measurable over a set  $E$  of finite measure with  $\int_E f = 0$ . Show that  $f = 0$  a.e.
4. Let  $g$  be integrable over  $[a, b]$ , and define  $f(x) = \int_a^x g, \forall x \in [a, b]$ . Show that  $f$  is absolutely continuous on  $[a, b]$ .

5. Define

$$f(x) = \begin{cases} 0 & x \text{ rational} \\ 1/a & x \text{ irrational and } a \text{ is the first non-zero integer in the decimal expansion of } x. \end{cases}$$

For example, if  $x = 0.002439\dots$ , then  $a = 2$ .

- (a) Prove that  $f$  is measurable.
  - (b) Find  $\int_0^1 f$  if the integral exists.
6. Suppose that  $f \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ .
- (a) Show that  $f \in L^p(\mathbb{R})$  for any  $1 < p < \infty$ .
  - (b) Show that  $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$ .

**MTH 636:** Provide complete solutions to **only** 7 of the 8 problems. \* is extra-credit.

1. Find the set of points  $z$  at which  $f(z) = |z^4|$  is differentiable. Justify your answer.
2. Assume  $R > 0$ ,  $M \geq 0$ . Suppose  $f(z)$  is analytic on  $|z| \leq R$ ,  $f(0) = 0$  and  $|f(z)| \leq M$  for  $|z| \leq R$ . Prove Schwarz's lemma, that is,

$$|f(z)| \leq \frac{M}{R}|z|.$$

3. Show that  $8e$ ,  $e = 2.718\dots$ , is an upper bound for  $|\int_C z^4 e^z dz|$ , where  $C$  is the straight line contour joining the points  $1 + i$  and  $1 - i$ .
4. If  $f$  is analytic for  $|z| \leq 1$  then show that

(a)

$$\frac{1}{\pi} \int_{|z| \leq 1} f(x + iy) dx dy = f(0).$$

\*(b) Show also that (Gauss' mean value theorem)

$$\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta = f(0).$$

5. If  $f(z) = g(z)/h(z)$ , where  $g$  and  $h$  are analytic at  $z = \alpha$ , but  $h$  has a simple zero at  $z = \alpha$ , then show that the residue of  $f$  at  $\alpha$  is given by

$$a_{-1} = \frac{g(\alpha)}{h'(\alpha)}.$$

6. Find the Laurent series expansion of

$$f(z) = \frac{2}{(z+3)(z+1)}$$

in the annulus  $1 < |z| < 3$ .

7. Use Rouché's Theorem to show that all zeros of the complex-valued polynomial

$$p(z) = z^7 - 5z^3 + 12$$

lie between the circles  $|z| = 1$  and  $|z| = 2$ .

8. Let  $m > 0$ . Show that

$$\int_0^{+\infty} \frac{\cos mx}{x^2 + 1} dx = \frac{\pi}{2} e^{-m}.$$