Analysis Qualifying Exam **January 4, 1997, 8:00-12:00**

Section A Answer 10 of the 12.

1. Consider the sequence of functions $\{f_n(x)\}$ defined by
$$f_n(x) = \frac{2nx}{1+n^2+x^2}.$$
   (a) Determine with proof the pointwise limit, f(x), of $\{f_n(x)\}$ and whether this convergence is uniform on $[-1,1]$.

   (b) Define $F_n = \int_{-1}^{1} f_n(t)\,dt$. Does $\lim_{n\to\infty} F_n = \int_{-1}^{1} f(t)\,dt$? If your answer is yes, is it necessary that the convergence of $\{f_n(x)\}$ be uniform? Provide proof, citing the appropriate theorems.

2. Suppose the function f is defined at (x_0, y_0) and both $\partial f/\partial x$ and $\partial f/\partial y$ exist at (x_0, y_0). Show that
$$\lim_{x\to x_0} f(x, y_0) = \lim_{x\to y_0} f(x_0, y) = f(x_0, y_0).$$
Does this imply that f is continuous at (x_0, y_0)? Provide proof of your answer.

3. Let V be an open subset of $\mathbb{R}^2$ containing $\vec{\mathbf{u}}_0 = (x_0, y_0)$. Suppose $F : V \to \mathbb{R}^2$ is continuously differentiable with a nonzero Jacobian, $J_F(x,y)$ on V. Let $B_r = \{\vec{\mathbf{u}} \in \mathbb{R}^2 : \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|_2 \le r\}$, where $r > 0$ is sufficiently small so that $B_r \subset V$.

   (a) Prove that
$$\lim_{r\to 0} \frac{A(F(B_r))}{\pi r^2} = |J_F(x_0, y_0)|.$$
   Here $A(F(B_r))$ is the area of the region $F(B_r)$.

   (b) Verify part (a) for $V = \{\vec{\mathbf{u}} \in \mathbb{R}^2 : \|\vec{\mathbf{u}}\|_2 < 2\}$, $F(x,y) = (4x+y, -x^2+y)$, and $\vec{\mathbf{u}}_0 = (0,0)$. (Hint: If R is a simply connected region in $\mathbb{R}^2$ with a smooth boundary C oriented positively, use Green's Theorem to show $A(R) = \frac{1}{2}\oint_C x\,dy - y\,dx$).

4. Let $f(x)$ be continuous and bounded on $[1,\infty)$ and suppose that
$$\int_1^\infty f(x)x^n\,dx = 0 \text{ for } n = -2, -3, \ldots.$$
Does it follow that $f(x) \equiv 0$? Justify your answer. (Hint: $x = 1/u$.)

5. A sequence $f_n \in L^2[a,b]$ is said to converge weakly to f if and only if $\int f_n g \to \int fg$ for all $g \in L^2[a,b]$. Prove:

   (a) If $f_n \to f$ in norm in $L^2[a,b]$, then $f_n \to f$ weakly.

   (b) If $f_n \to f$ weakly, $f \in L^2[a,b]$, and $\|f_n\|_2 \to \|f\|_2$, then $f_n \to f$ in the norm of $L^2[a,b]$.

   (c) The converse of (a) may fail.

6. (a) Suppose $f \in L^1[a,b]$. Show that the function F defined by $F(x) = \int_a^x f(t)\,dt$ is continuous and of bounded variation on $[a,b]$.

(b) Given $F(x) = x^{3/4}$ on $[0,1]$, decide whether F is absolutely continuous on $[0,1]$. Give a proof or a counterexample. You may quote related theorems.

7. Let $E \subset \mathbb{R}$ be a Borel set.

   (a) Show $m(E) = \inf\{m(U) : E \subset U, U \text{ open}\}$. Here m is Lebesgue measure on $\mathbb{R}$.

   (b) Show there exists a sequence of open sets $\{U_n\}_{n=1}^{\infty}$ such that $U_n \supset U_{n+1}$ and $\lim_{n\to\infty} m(U_n) = m(E)$.

8. Suppose f is a measurable function on $[0,1]$.

   (a) Must $|f|$ be measurable? Give a proof or a counterexample.

   (b) If $\int_0^1 |f| = 0$, must $f = 0$ a.e.? Give a proof or a counterexample.

9. (a) Define what is meant by saying that f is analytic at a point a in a domain G? Is this the same as saying F is differentiable at a?

   (b) State the Cauchy-Riemann equations as they apply to $f(z) = u(x,y) + i\,v(x,y)$, where $z = x + i\,y$.

   (c) Find all z at which $f(z) = x^3y + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$ is differentiable. For which z is f analytic?

10. State the definitions of all the different kinds of singularities for a complex-valued function on a region. Give an example of each kind.

11. Use the Cauchy Integral Formula for circular paths to prove that f is analytic in a disk of radius R centered at $z = a$, then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ for $|z - a| < R$.

12. Find the Laurent series of $\dfrac{1}{(z-1)(z+2)}$.

    (a) for $1 < |z| < 2$,

    (b) for $|z| > 2$.

Section B Select one of the two parts

Part 1 Answer 2 of the three

1. Let $\{f_n\}$ be a sequence of measurable functions on a measure space (X, Σ, μ) such that there is a measurable function f on X with the property that for every $\epsilon > 0$ there is a positive integer N and a measurable set E such that $\mu(E) < \epsilon$ and $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and all $x \notin E$. (That is, $\{f_n\}$ converges to f in measure.) Show that there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$, μ almost everywhere. Show that the converse is also true if $\mu(X) < \infty$.

2. (a) Let f be an integrable function on a measure space (X, Σ, μ). Show that for every $\epsilon > 0$ there exists a measurable function g on X such that $\mu\{x \in X : g(x) \neq 0\} < \infty$, and $\int |f - g|\, d\mu < \epsilon$.

   (b) Show that the g in part (a) can be chose to be a simple function.

   (c) If the measure space is the real line with Lebesgue measure, show that the g in part (a) can be chosen to be a step function.

3. On the Borel σ-algebra in $[1,\infty)$, let $\mu(E) = \int_E x^{-1}\, dm$. Here m is Lebesgue measure. Show that

   (a) $\mu \ll m$, $m \ll \mu$.

   (b) $1 \le p < \infty$, $L^p(m)$ is a proper subset of $L^p(\mu)$.

   (c) $L^\infty(m) = L^\infty(\mu)$.

Part 2 Answer 2 of the three

1. Find the function $u(x,y)$ that is harmonic in the unit disk $|z| < 1$ and takes on the boundary values $u(\cos\theta, \sin\theta) = \frac{\theta}{2}$, for $-\pi < \theta \le \pi$.

2. Locate the branch points, suggest how the plane should be cut and pasted to form a Riemann surface for the multi-valued function $[(z-1)(z+2)]^{2/3}$. Specify an analytic branch of the function.

3. If possible, construct an entire function which takes on the values only in the right half-plane. If this is not possible, explain.