Analysis Qualifying Exam
March 20, 2020

MTH 636: Provide complete solutions to **only** 5 of the 6 problems.

1. Let f be a holomorphic function on a connected open set $U \subset \mathbb{C}$. Prove that if $\text{Re}f + \text{Im}f$ is constant, then f is also constant.

2. Let f be a holomorphic function on the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and suppose that $|f(z)| \leq e^{\text{Re} z}$ for all $z \in \mathbb{C}^*$. Show that $f(z) = ce^z$ for some constant $c \in \mathbb{C}$. What can we conclude about the absolute value of the constant c?

   Hint: Use the Riemann removable singularity theorem and Liouville's theorem.

3. Find all points in the complex plane at which the function
$$f(z) = \overline{z}^2 \cos z$$
   is complex-differentiable.

4. (a) (3 points) Find the Laurent series expansion of the function
$$f(z) = \frac{1}{z(z-1)(2-z)}$$
   in the punctured disc $\{0 < |z| < 1\}$.

   (b) (3 points) Find and classify all isolated singularities of the function $g(z) = \csc\left(\frac{\pi}{z}\right)$. Comment on the nature of the singularity of g at $z = 0$.

   (c) (4 points) Show that there is a sequence of complex numbers $\{a_n\} \subset \mathbb{C}$ such that each $a_n \neq 1$, we have $\lim_{n\to\infty} a_n = 1$ and
$$\lim_{n\to\infty} \sin\left(\frac{1}{1-a_n}\right) = 2020.$$

   Hint: for part (c), consider the type of singularity of the function $\sin\left(\frac{1}{1-z}\right)$ at $z = 1$.

5. Using Rouché's theorem, determine the number of zeroes of the function $f(z) = 4 + 9z + 3z^2 + z^4$ in the annulus $\{1 < |z| < 3\}$ centered at 0, with inner radius 1 and outer radius 3.

6. Let $a > 0$ be a real number. Use the Residue Theorem to evaluate the integral
$$\int_{-\infty}^{\infty} \frac{dx}{a^4 + x^4}.$$
   Make sure to express your answer as a real number.

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

1. Define the function f on $(0, \infty)$ by setting

$$f(x) = \frac{\sin(x)}{x}, \text{ for } x > 0.$$

Prove that f is measurable, but its Lebesgue integral $\displaystyle\int_{(0,\infty)} f$ does not exist.

2. (a) (1 point) For each real number t show that

$$\frac{t}{1+t^2} \leq \frac{1}{2}.$$

(b) (2 points) For a positive integer n let $f_n : [0,1] \to \mathbb{R}$ be defined as

$$f_n(x) = \frac{n\sqrt{x}}{1+n^2x^2}, \quad x \in [0,1].$$

Show that for each positive integer n and each $x \in (0,1]$ we have

$$0 \leq f_n(x) \leq \frac{1}{2\sqrt{x}}.$$

(c) (3 points) Does the sequence of functions $\{f_n\}$ converge uniformly to a limit on $[0,1]$? Justify your response.

(d) (4 points) Compute the limit $\displaystyle\lim_{n\to\infty} \int_{[0,1]} f_n$.

Hint for part (d): Use the Dominated Convergence Theorem.

3. For each of the following, either give an example of the object described, or state that it does not exist. Justify that your example indeed works, and if an example does not exist, justify your statement that such an example does not exist.

(a) (3 points) A nonmeasurable function $f : \mathbb{R} \to \mathbb{R}$ such that the function $\cos(f)$ is measurable.

(b) (3 points) A function which belongs to $L^5([0,1])$ but not to $L^2([0,1])$.

(c) (4 points) A sequence of continuous functions $\{f_n\}$ on $\mathbb{R}$ such that $\int_{\mathbb{R}} f_n = 1$ for each n, and $f_n \to 0$ uniformly.

4. For each of the following, either give an example of the object described, or state that it does not exist. Justify that your example indeed works, and if an example does not exist, justify your statement that such an example does not exist.

   (a) (3 points) A nonmeasurable function $f : \mathbb{R} \to \mathbb{R}$ such that the function e^f is measurable.

   (b) (3 points) A continuous function which belongs to $L^1(\mathbb{R})$ but not to $L^\infty(\mathbb{R})$.

   (c) (4 points) A function f on $(0, 1)$ which is differentiable at each point such that the derivative f' is not a measurable function.

5. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}$ that converges pointwise on $\mathbb{R}$ to f. Suppose that $f \in L^1(\mathbb{R})$, and $\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$. Then show that for each measurable subset E of $\mathbb{R}$, the limit $\lim_{n\to\infty} \int_E f_n$ exists, and is equal to $\int_E f$.

6. Let E be a measurable subset of $\mathbb{R}$ and let $1 \le p < \infty$. Suppose that a sequence $\{f_n\} \subset L^p(E)$ converges pointwise a.e. to f, and $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$. Prove that $\lim_{n\to\infty} \|f_n - f\|_p = 0$.
   Hint: You may use without proof that for $1 \le p < \infty$ and any real a, b, one has
   $$|a - b|^p \le 2^p(|a|^p + |b|^p).$$