Solution to problem # 659

PROBLEM: a) Prove that for any five points P, A, B, C, and D in the plane,

$$PA + PB + PC + PD > 0.33(AB + AC + AD + BC + BD + CD).$$

b) Is it possible to find five points P, A, B, C, and D in the plane such that

$$PA + PB + PC + PD < 0.34(AB + AC + AD + BC + BD + CD)?$$

SOLUTION: (A) For any three points X, Y, Z in the plane we have $XY + YZ \geq XZ$ with equality if and only if the point Y is on the segment XY. Therefore we have

$$\begin{aligned} &PA + PB \geq AB, \qquad PA + PC \geq AC, \qquad PA + PD \geq AD \\ &PB + PC \geq BC, \qquad PB + PD \geq BD, \qquad PC + PD \geq CD. \end{aligned}$$

Adding the six inequalities above, we get

$$3(PA + PB + PC + PD) \geq AB + AC + AD + BC + BD + CD,$$

with equality if and only if there is equality in each of the six inequalities added, i.e., P is located on each of the segments AB, AC, AD, BC, BD and CD. Since $\frac{1}{3} = \frac{33}{99} > \frac{33}{100} = 0.33$, we see that

$$\begin{aligned} PA + PB + PC + PD &\geq \frac{1}{3}(AB + AC + AD + BC + BD + CD) \\ &> 0.33(AB + AC + AD + BC + BD + CD). \end{aligned}$$

(B) Motivated by the condition for equality above, consider the points on the x-axis with coordinates (where $h > 0$):

$$P = (0, 0), \quad A = (0, -h), \quad B = (0, -2h), \quad C = (0, -3h), \quad D = (1, 0).$$

Then by a direct calculation:

$$\frac{PA + PB + PC + PD}{AB + AC + AD + BC + BD + CD} = \frac{1 + 6h}{3 + 10h} = f(h).$$

If $h = 0$ the four points P, A, B, C coincide, and the ratio above is $f(0) = \frac{1}{3}$. For $h > 0$, the five points are distinct and we see that $f(1) = \frac{7}{13} > \frac{1}{2} = 0.5$. It follows from the continuity of the function f and the intermediate value theorem that there are values of h with $0 < h < 1$ such that $f(h) < 0.34$.

A numerical calculation shows that for $0 < h < \frac{1}{130}$, we have $f(h) < 0.34$.