# Solution to problem # 661

**PROBLEM:** Find all angles $\alpha$ such that $\cos(\alpha)$ is a rational number, but $\cos(2023\alpha)$ is an irrational number.

If such an $\alpha$ exists, give an explicit example.

If no such $\alpha$ exists, give a proof of nonexistence.

**SOLUTION**: There is no such $\alpha$. In fact if $\cos(\alpha)$ is rational, then $\cos(n\alpha)$ is rational for each positive integer $n$. To see this, proceed by induction on $n$. The base case $n = 1$ is true by assumption. Note that the statement is also true for $n = 0$ because $\cos(0) = 1$ is rational.

Assume that $\cos(m\alpha)$ is rational for non-negative integers $m \leq n$. By the addition formulas for cosine,

$$\begin{aligned}\cos\left((n+1)\alpha\right) &= \cos(n\alpha)\cos(\alpha) - \sin(n\alpha)\sin(\alpha),\\ \cos\left((n-1)\alpha\right) &= \cos(n\alpha)\cos(\alpha) + \sin(n\alpha)\sin(\alpha).\end{aligned}$$

We manipulate these two equations to find that

$$\cos\left((n+1)\alpha\right) = 2\cos(n\alpha)\cos(\alpha) - \cos\left((n-1)\alpha\right).$$

All of the terms on the right-hand side are rational, so $\cos((n+1)\alpha)$ is rational.