Solution to problem # 662

PROBLEM: Let $a_0, a_1, \ldots, a_n$ be real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0.$$

Show that the equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

has at least one real root.

SOLUTION: Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. Then

$$\int_0^1 f(x)dx = \frac{a_0}{1} + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0.$$

It follows that on the interval $[0,1]$ the function f is either identically zero, or must assume both positive and negative values. In the latter case, since f is continuous it must vanish in some point of the interval.