CENTRAL MICHIGAN UNIVERSITY
DEPARTMENT OF MATHEMATICS
ANALYSIS QUALIFYING EXAM
JANUARY 19, 2024

Each question of this exam is worth 10 points. For questions divided into multiple sub-questions, the points for each part are indicated.

1. Instructions

(1) The exam has 6 problems from Complex Analysis (MTH 636), and 6 problems from Real Analysis (MTH 632).

(2) You are required to complete 5 problems from the Complex Analysis (MTH 636) part, and another 5 problems from the Real Analysis (MTH 632) part.

(3) In either section, if you attempt solutions to all 6 problems, then only the first 5 problems will be graded.

(4) Begin each problem on a separate sheet of paper, and clearly write the problem number and your name before beginning your solution.

(5) No calculators or other electronic devices are allowed.

(6) No questions may be asked during the exam. If a problem appears ambiguous to you, interpret it in a way that makes sense to you, but not in a way that makes it trivial.

(7) Give proper mathematical justification for all your statements.

(8) Throughout the exam, if a question has multiple parts, you may assume any previous part as a true statement in order to answer the consecutive parts whenever needed.

MTH 636: Provide complete solutions to **only** 5 of the 6 problems.

(1) Let $C = \{z : |z| = 1\}$ denote the unit circle oriented counterclockwise. Compute each of the following three integrals:

(a) **(3 points)** $\displaystyle\int_C \frac{\sin(z^2)}{z^3 - 6} dz$

(b) **(3 points)** $\displaystyle\int_C \frac{\sin(z^2)}{z^3} dz$

(c) **(4 points)** $\displaystyle\int_C \frac{\sin(z^2)}{(\overline{z})^3} dz$

(2) Let $\Omega = \mathbb{C} \setminus \{0\}$ be the complex plane with the origin removed. Find all holomorphic functions on Ω such that $f(1) = 1$ and

$$|f(z)| \leq \frac{20}{|z|^{24}} \text{ for all } z \in \Omega.$$

(3) Determine all isolated singularities of the function

$$f(z) = z^2 \cos\left(\frac{1}{z}\right) + \frac{z(z-1)^2(z-2)}{\sin^2(\pi z)}$$

and classify each singularity as removable, pole, or essential. Determine the order of each pole.

(4) Let f and g be holomorphic functions on the unit disk $D = \{z : |z| < 1\}$ such that the product fg vanishes identically on D. Show that one of the functions f and g vanishes identically on D.

(5) Find the Laurent expansion of the function

$$f(z) = \frac{2z - 5}{z^2 - 5z + 4}$$

on the annulus $\{z : 1 < |z| < 4\}$ (centered at $z = 0$).

(6) Compute the integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - a^2}{1 - 2a\cos\theta + a^2} d\theta,$$

where $0 \leq a < 1$ is a real number (hint: express the integral as a line integral around the unit circle).

MTH 632: Provide complete solutions to **only** 5 of the 6 problems.

(1) Argue why the following sets are Lebesgue measurable, and find their measures (with justification).

(a) **(3 points)** $[0,1] \cap \mathbb{Q}$.

(b) **(3 points)** $[\pi, 2\pi] \cap (\mathbb{R} \setminus \mathbb{Q})$.

(c) **(4 points)** $\bigcup_{n=0}^{\infty} \left[n, n + \frac{1}{2^n}\right]$.

(2) Let $(f_n \colon E \to \mathbb{R})$ be a **decreasing** sequence of **positive** measurable functions.

(a) **(5 points)** Prove that there is a **measurable** function $f \colon E \to \mathbb{R}$ such that $f_n \to f$ pointwise.

(b) **(5 points)** Suppose that f_1 is integrable. Prove that

$$\lim_{n \to \infty} \int_E f_n = \int_E f.$$

(3) Recall that a sequence of functions $(f_n \colon E \to \mathbb{R})$ converges uniformly to $f \colon E \to \mathbb{R}$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $x \in E$ and $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Construct a sequence of measurable functions $(f_n \colon \mathbb{R} \to \mathbb{R})$ that converges pointwise a.e. to a function f such that for any set U on which $(f_n|_U)$ converges uniformly to $f|_U$,

$$m(\mathbb{R} \setminus U) > 1.$$

(Recall that $f|_A$ is the restriction of f to A, sending each $a \in A$ to $f(a)$.)

(4) Let E be a measurable subset of $\mathbb{R}$ and let $f \colon E \to \mathbb{R}$ be integrable. For each n let $E_n \subset E$ be a measurable subset, and suppose that

$$E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$$

Prove that

$$\int_{\bigcup_{n=1}^{\infty} E_n} f = \lim_{n \to \infty} \int_{E_n} f.$$

(5) Let $f \colon [-1,1] \to \mathbb{R}$ be defined by $f(x) = x^{2024}$.

(a) **(5 points)** Fix integers $0 < k < n$. Let $P = \{a_0, \dots, a_k, a_{k+1}, \dots, a_n\}$ denote a partition of $[-1,1]$ where

$$\begin{aligned} a_0 &= -1, \\ a_k &\leq 0 < a_{k+1}, \text{ and} \\ a_n &= 1. \end{aligned}$$

Compute the variation of f with respect to P to show that it is given by

$$V(f,P) = 2 - 2\min(a_k^{2024}, a_{k+1}^{2024}).$$

(b) **(5 points)** Show that the total variation of f is $TV(f) = 2$.

(6) Suppose $1 \leq p < \infty$ and q is the conjugate of p. Prove that if $f \in L^p(E)$ then

$$||f||_p = \max\left\{ \int_E fg \;\middle|\; g \in L^q(E) \text{ and } ||g||^q \leq 1 \right\}.$$