Algebra Qualifying Exam

August 24, 1996

There are four pages. A grade of 135 or more out of 180 will guarantee passing.

Part A (60 points)

Do any three (3) of parts (1) - (4). Each problem is worth 20 points. If you do all four parts, only the first three will be graded.

1. Let T be a linear transformation from a finite-dimensional vector space V to a vector space W. Let B denote a basis for V.
   a. Show that T(B) is a basis for the range of T if and only if T is one-to-one.
   b. Does this same result hold if V is infinite-dimensional? Justify your answer.

2. a. Give an argument to show that any 5 vectors in $\mathbf{R}^4$ must be linearly dependent. (Do not simply say that it is because $\mathbf{R}^4$ has dimension 4.)
   b. Let $S = \{v_1, v_2, \ldots, v_n\}$ be a basis for the vector space V. Show that any set of more than n vectors in V is linearly dependent.
   c. Prove that any two bases for a finite-dimensional vector space must have the same number of elements.

3. Suppose $\underline{\alpha} = \{u_1, u_2, \ldots, u_n\}$ and $\underline{\beta} = \{v_1, v_2, \ldots, v_n\}$ are two bases for a vector space V.
   a. Assume that A and B are matrices representing the same linear transformation T with respect to the bases $\underline{\alpha}$ and $\underline{\beta}$ respectively. Show that A and B are similar.
   b. Decide if $A = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}$ is similar to a diagonal matrix. Justify your answer.
   c. Can you write $A = UDU^T$ where $UU^T = I$? Justify your answer.

4. Suppose V is a vector space of dimension n over a field F. Let $L(V)$ denote the set of all linear transformations on V.
   a. Is $L(V)$ a vector space, and if so, what is its dimension? (Justify your answers.)
   b. Show that if $T \in L(V)$, there is a nonconstant polynomial f of degree less than or equal to n^2 so that $f(T) = 0$.
   c. Define the notions of minimal polynomial and characteristic polynomial. Discuss the relationships between the two notions, including a statement of any appropriate theorems.
   d. Find the characteristic and minimal polynomial of the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Part B (60 points)

Do any <u>four</u> (4) of parts (1) - (5). Each problem is worth 15 points. If you do all five parts, only the first four will be graded.

1. Let $GL(n,\mathbf{R})$ denote the set of all n x n matrices over **R** whose determinant is different from zero. Set $SL(n,\mathbf{R}) = \{A: A \in GL(n,\mathbf{R}) \text{ and } \det A = 1\}$.
   a. Prove that $GL(n,\mathbf{R})$ is a group under matrix multiplication.
   b. Prove that $SL(n,\mathbf{R})$ is a normal subgroup of $GL(n,\mathbf{R})$.
   c. Prove that $GL(n,\mathbf{R})/SL(n,\mathbf{R})$ is isomorphic to the group of non-zero elements of **R** under multiplication.

2. a. State precisely the full content of the Sylow Theorems for a finite group G whose order is divisible by a prime p.
   b. Prove that a group G of order 126 must contain a normal subgroup of order 7.
   c. Prove that a group of order 1,000 cannot be a simple group.

3. a. Let G be a group in which the square of every element is the identity. Prove that G is abelian.
   b. Prove that a group G is abelian if and only if the function $f:G \to G$ defined by $f(x) = x^{-1}$ is a homomorphism.

4. Let $G = \langle a \rangle$ be a cyclic group of order n. Prove that $G = \langle a^k \rangle$ if and only if gcd (k,n) = 1. Conclude that an integer k is a generator of $\mathbf{Z}_n$ if and only if gcd (k,n) = 1.

5. Let f: $G \to H$ be a non-trivial homomorphism, i.e., f does not send every element of G onto the identity of H. If G is simple, prove that f is one-to-one.

Part C (60 points)

Do any <u>five</u> (5) of parts (1) - (7). Each problem is worth 12 points. If you do more than five parts, only the first five numerically will be graded.

1. Let R be a ring whose additive group (R,+) is cyclic. Show that R is commutative.

2. Let f: $R \to S$ be a ring homomorphism. A ring homomorphism F: $R[x] \to S[x]$ is defined by

$$F(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n) = f(a_0) + f(a_1)x + f(a_2)x^2 + \ldots + f(a_n)x^n$$

How are the kernel of F (ker F) and the image of F (im F) related to ker f and im f?

3. Let R be a ring with identity.
(3) a. Define the characteristic of R.
(3) b. Give an example of a ring with characteristic 6. (You need not offer proof.)
(6) c. Prove that the characteristic of an integral domain is zero or a prime number.

4. Let R be a ring and $I \neq R$ an ideal of R.
(3) a. (Complete) I is a maximal ideal of R if and only if R/I is __________.
(9) b. Prove your statement in (a).

5. Let $K^{n \times n}$ be the ring of all n x n matrices over a field K. For $1 \leq k \leq n$, let $U_k = \{[\mathbf{0} \ldots \underset{\substack{\uparrow \\ \text{k-th column}}}{\mathbf{v}} \ldots \mathbf{0}]: \mathbf{0}$ is the zero vector in K^n and $\mathbf{v} \in K^n\}$,

   i.e., U_k is the set of all matrices in $K^{n \times n}$ which are zero everywhere except possibly in the k-th column.

   (6) a. Show that U_k is a subring of $K^{n \times n}$.

   (6) b. Prove or disprove that U_k is an ideal of $K^{n \times n}$.

6. A topic in ring theory is divisibility (or factorization) in integral domains. State and prove a result about divisibility (factorization) in integral domains. Define all the terms in your statement and proof that deal with divisibility (factorization). (You need not define other ring theory terms.)

7. a. Is *L*(V) of Part A, question $\cancel{3}$ 4 an R module? If so, for what R and what operations?

   b. Is GL(n,**R**) (see Part B, question 1) an R module? If so, for what R and what operations?

   c. If you answered no to both (a) and (b), give an example of an R module M. Specify what R, M, and all relevant operations are.