Algebra Qualifying Examination

January 22, 1999

There are three parts to this exam. Part A covers linear algebra, part B covers group theory, and part C covers ring theory. At various points in the exam, you are asked to do a certain number of questions from a larger number of questions. If you do more than the stated number of parts, only the first ones numerically will be scored.

Each problem is worth 10 points. Some of the problems have multiple parts. In some instances, the point values for the different parts have been indicated. If there is no such indication, then the parts of the problem all have equal value.

The letters $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{C}$ denote the integers, the real numbers, and the complex numbers respectively.

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Part A. Do any four of the five problems in this part.

1. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation that has matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

with respect to the standard basis.

   (a) (4 points) Find the matrix $[T]_\beta^\beta$ of the linear transformation with respect to the basis

$$\beta = \left\{ w_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

   (b) (3 points) Find a basis for the nullspace of T. (The *nullspace* of T is also called the *kernel* of T.)

   (c) (3 points) Find a basis for the range of T. (The *range* of T is also called the *image* of T.)

2. **Prove:** If A is an $n \times n$ matrix over the complex numbers and $n > 1$, then A is the sum of two singular matrices. **Hint:** First prove it for upper triangular matrices. After that, Schur's theorem is useful.

3. Let J be the $n \times n$ matrix with all entries 1, and let I be the usual $n \times n$ identity matrix.

   (a) Find all eigenvalues of J. For each eigenvalue, find its geometric multiplicity and its algebraic multiplicity.

   (b) Express the determinant of the matrix $2I + J$ as a function of n.

4. A matrix A is **positive definite** if A is Hermitian and $\langle Ax, x \rangle > 0$ for all non-zero x.

   (a) Prove: A is positive definite if and only if all eigenvalues of A are positive.

   (b) Prove: A is positive definite if and only if there is some invertible matrix B such that $A = B^*B$.

5. Which of the following pairs of matrices are similar? Justify your answers.

   (a) (3 points)

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}.$$

   (b) (3 points)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   (c) (4 points)

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Part B Do any four of the five problems in this part.

1. Suppose that H is a subgroup of G and that a and b are elements of G.
   (a) Prove that if H is normal and $ab \in H$ then $ba \in H$.
   (b) Is part(a) true if the restriction that H is normal is removed? Justify your answer.
2. Let S_n denote the symmetric group on n letters, and let A_n denote the alternating group on n letters; i.e., A_n is the subgroup of even permutations of S_n.
   (a) Does S_4 have a subgroup of order 6? Justify your answer.
   (b) Does A_4 have a subgroup of order 6? Justify your answer.
3. Suppose that H and K are subgroups of a group G. Define
$$HK = \{hk : h \in H, k \in K\}.$$
   Prove that HK is a subgroup of G if and only if $HK = KH$.
4. Let $\mathbf{T}$ denote the circle group; i.e., $\mathbf{T}$ is the set of complex numbers of modulus 1 under the operation of multiplication. In other words, $\mathbf{T} = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$. Prove that $\mathbf{T}$ is isomorphic to $\mathbf{R}/\mathbf{Z}$.
5. Consider the group A_5, the group of even permutations on five letters.
   (a) (2 points) How many subgroups of order 3 does A_5 have?
   (b) (2 points) List all of the elements in one of the subgroups of order 3.
   (c) (6 points) Let H be the group you gave in part (b). How many conjugates does H have in A_5?

Part C This section of the exam is divided into two parts.

Part C1 Do any two of the three questions in this part.

1. Recall that a Euclidean domain is a commutative integral domain E with unity such that there exists a function $\phi : E \to \mathbf{Z}$ satisfying the following axioms:

   (i) If $a, b \in E^* = E - \{0\}$, and b divides a, then $\phi(b) \leq \phi(a)$.

   (ii) For each pair of elements $a, b \in E$, $b \neq 0$, there exist unique elements q and r in E such that $a = bq + r$ and $\phi(r) < \phi(b)$.

   Suppose that E is a Euclidean domain and that $c, d \in E^*$. Which of the following is always correct? $\phi(c) \leq \phi(cd)$ or $\phi(c) \geq \phi(cd)$? Justify your answer.

2. Consider the ring $M_n(\mathbf{Z})$ of $n \times n$ matrices with integer entries. **It can be shown that the irreducible elements of this ring are precisely those matrices with determinant that is prime in Z.**

   (a) (3 points) Give an example of an irreducible element in $M_3(\mathbf{Z})$.

   (b) (3 points) For an element of a ring to be irreducible, it must not be a unit. In the above statement in bold, why are the matrices in $M_n(\mathbf{Z})$ with prime determinants not units?

   (c) (4 points) Give an example of a zero divisor in $M_3(\mathbf{Z})$.

3. Suppose that I and J are ideals of a commutative ring R. Define the set $(I : J)$ by

$$(I : J) = \{x \in R : xJ \subseteq I\}.$$

   (a) (2 points) In the ring $\mathbf{Z}$, describe $((2) : (6))$.

   (b) (2 points) In the ring $\mathbf{Z}$, describe $((6) : (2))$.

   (c) (6 points) Prove (in the general case) that $(I : J)$ is an ideal of R.

Part C2 Do any two of the three questions in this part.

1. Suppose that I and J are two ideals of a ring R. Prove: If I is not a subset of J and J is not a subset of I, then the ideal $I \cap J$ is not prime.

2. Suppose that S is a subring and that I is an ideal of the ring R. Prove that if $S \cap I = \{0\}$, then S is isomorphic to a subring of the quotient ring R/I. (**Hint:** Utilize the mapping $f(a) = a + I$, where $a \in S$.)

3. Suppose that f is a homomorphism from a ring R to a ring S. Prove or disprove any one (1) of the following statements.

   (a) If M is a maximal ideal of R, then $f(M)$ is a maximal ideal of S.

   (b) If N is a nilpotent ideal of R, then $f(N)$ is a nilpotent ideal of S.

   (c) If I is maximal ideal of S, then $f^{-1}(I)$ is a maximal ideal of R.