PH.D. QUALIFYING EXAMINATION – THEORETICAL STATISTICS
Time: 8:00 AM – 12:00 Noon, August 24, 2023

General Instructions

- There are two parts in this exam: STA 584 and STA 684. You are to answer all questions. The raw score for each part will be converted to its percentage.
- Write on one side only. Begin each subpart on a new sheet with the problem number clearly labeled. You must show all your work and justifications correctly and completely to receive full credits. Partial credits may be given for partially correct solutions.
- For each problem/subproblem, hand in only the answer that you want to be graded. Crossed-out work will be ignored. Failure to follow this instruction for a problem will result in a zero score for the problem.
- If a theorem is applied, you must state the theorem, identify its assumption(s) and conclusion(s), and justify why it is applicable. New notations must be defined before use. You need to specify the support of the distribution in your answer.
- When finished, please collate all pages based on problem labels and then number the pages accordingly. Hand in also the exam paper.

By signing below, I hereby acknowledge that I have completely read and fully understand the instructions.

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Signature

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Printed Name

Part A - STA 584

1. (12 *points*, 3 *each*) Answer all of the following questions.

1.a) The number of claims handled by an insurance company has a Poisson distribution with a mean of 5. What is the probability that there will be 2 claims in 3 of the next 5 days? Assume that the numbers of claims on different days are independent.

1.b) A contractor purchases a shipment of 20 transistors. It is his policy to test 5 of these transistors and to keep the shipment only if at least 4 of the 5 are in working conditions. If the shipment contains 4 defective transistors, what is the probability that the shipment will be kept?

1.c) About 10% of the U.S. population has Type B blood. The American Red Cross will hold a blood drive at Gurnee Mills. What is the probability that the 10th donor is the first donor with Type B blood? How many blood donors should the American Red Cross expect to collect from until it gets a donor with Type B blood?

1.d) What are the mean and standard deviation of a random variable with moment generating function given by $(1-2t)^{-2}$ where $t < 1/2$.

2. (8 *points*, 3 *each* unless otherwise stated) Let X and Y be independent and identically distributed random variables with common probability density function (pdf) given by $f(x) = e^{-x}$, $x > 0$, zero elsewhere. Define

$$U = \frac{X}{X+Y} \quad \text{and} \quad W = X + Y.$$

2.a) Find the joint pdf of U and W.

2.b) Find the pdf of W. Identify the name of the random variable associated with the pdf.

2.c) Are U and W independent? Justify your answer. (2 *points*)

3. (22 *points*, 3 *each* unless otherwise stated) Let X and Y have the joint probability density function

$$f(x, y) = k \text{ for } -x < y < x,\ 0 < x < 1, \text{ zero elsewhere.}$$

3.a) Find the constant k. (2 *points*)

3.b) Find $P(2Y > X)$.

3.c) Find the marginal pdf of Y. Check whether your answer can serve as a pdf before proceeding.

3.d) Find conditional pdf of Y given $X = x$.

3.e) Derive the conditional mean of Y given $X = x$. (1 *point*)

3.f) Find the conditional variance of Y given $X = x$. (2 *points*)

3.g) Find the variance of Y.

3.h) Find the moment generating function of Y.

3.i) Find the correlation coefficient of X and Y. (2 *points*)

4. (13 *points*, 2 *each* unless otherwise stated) Let X and Y have the joint probability mass function (pmf) given by

$$f(x, y) = \frac{2x + y}{12} \text{ for } (x, y) = (0, 1), (0, 2), (1, 2), (1, 3), \text{ zero elsewhere.}$$

Define $U = Y - X$, $V = X + Y$, and $W = 3X - Y$.

4.a) What is the cumulative distribution function of X?

4.b) Find the expected value $E(X/Y)$.

4.c) Find the conditional variance of X given $Y = 2$.

4.d) Find the covariance between X and Y. (3 *points*)

4.e) Find the covariance between X and W.

4.f) Find the joint pmf of (U, V).

STA 684 Part

1. (10 *points*) Let $Y_1 \leq Y_2 \leq \ldots \leq Y_n$ be the order statistics of a random sample of size n from a distribution of the continuous type that has cdf $F(x)$ and pdf $f(x) = F'(x)$. Show that the limiting distribution of

   (1.a) (5 *points*) $U_n = n[1 - F(Y_n)]$ and $V_n = nF(Y_1)$ follow $Gamma(1,1)$.
   (1.b) (5 *points*) $T_n = nF(Y_2)$ is $Gamma(2,1)$ *and* $W_n = nF(Y_3)$ is $Gamma(3,1)$.

2. (25 *points*) Let $X_1, X_2, X_3,$ and X_4 are independent random variables such that $X_i = Gamma(\alpha_i, 1)$ for $i = 1,2,3,4$.

   (2.a) (10 *points*) Derive the joint density function of the new random variables $Y_1, Y_2, Y_3,$ and Y_4 such that
   $$Y_1 = \frac{X_1}{Y_4}, Y_2 = \frac{X_2}{Y_4}, Y_3 = \frac{X_3}{Y_4}, \text{and } Y_4 = X_1 + X_2 + X_3 + X_4.$$
   (2.b) (5 *points*) Using the pdf derived in Part (2.a), obtain the joint density of $Y_1, Y_2,$ and Y_3.
   (2.c) (10 *points*) Using the pdf derived in Part (2.b), obtain the conditional pdf $f(Y_1|Y_2, Y_3)$.

3. (10 *points*) Let $X_1, X_2, \ldots, X_n$ constitute a random sample from the following density function.
   $$f(x;\theta) = \begin{cases} ln\theta \cdot \theta^{1-x} & \text{for } 1 < x < \infty, \theta > 1 \\ 0 & \text{otherwise.} \end{cases}$$
   Let $g(\theta) = 1 + \frac{1}{ln\theta}$ and $\bar{X}$ be the sample mean. Show that

   (3.a) (2 *points*) the estimator $\bar{X}$ is a sufficient estimator for θ.
   (3.b) (8 *points*) the estimator $\bar{X}$ is a UMVUE estimator for $g(\theta)$.

4. (30 *points*) Consider a random sample $X_1, X_2, \ldots, X_n$ from the following pdf
   $$f(x) = \begin{cases} \frac{1}{2\rho}; & \theta - \rho \leq x \leq \theta + \rho, \quad \theta > 0, \ \rho > 0 \\ 0; & \text{otherwise} \end{cases}.$$

   (4.a) (7 *points*) Find the maximum likelihood estimators (mle), $\hat{\theta}$ and $\hat{\rho}$ for the parameters θ and ρ.
   (4.b) (8 *points*) Are these estimators $\hat{\theta}$ and $\hat{\rho}$ unbiased? Justify your answer.
   (4.c) (5 *points*) If you know the value of parameter θ, show that the mle of ρ is unique.
   (4.d) (5 *points*) If you know the value of parameter ρ, show that the mle of θ is not unique.
   (4.e) (5 *points*) Show that the likelihood ratio statistics λ- for testing $H_0: \theta = \theta_0$ against $H_a: \theta \neq \theta_0$ can be written as
   $$2\lambda^{1/n} = \frac{\max_{1 \leq i \leq n} x_i - \min_{1 \leq i \leq n} x_i}{max\left\{\theta_0 - \min_{1 \leq i \leq n} x_i, \max_{1 \leq i \leq n} x_i - \theta_0\right\}}$$