General Instruction: This is a closed exam with no any textbook, notes and electronic devices. The exam lasts 180 minutes. Answer all questions. Print your answer and number the pages and number the problems on provided exam papers. Write on one side only. Please use only black pens or pencils.

Notation of Multiple Linear Regression Model:

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad X = \begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1k} \\ 1 & x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & \vdots & & \vdots & \\ 1 & x_{n1} & x_{n2} & \dots & x_{nk} \end{bmatrix} \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$E(\boldsymbol{\varepsilon}) = \mathbf{0} \quad \boldsymbol{\varepsilon} = N(\mathbf{0}, \sigma^2\boldsymbol{I}) \qquad Cov(\boldsymbol{\varepsilon}) = \sigma^2\boldsymbol{I}$$

1. Given the model 1, $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon} \sim \boldsymbol{N}\left(\boldsymbol{0}, \sigma^2\boldsymbol{I}\right)$, $\boldsymbol{Y}_{n\times 1}$, $\boldsymbol{X}_{n\times(k+1)}$, $\boldsymbol{\beta}_{(k+1)\times 1}$, p= k+1 and $\boldsymbol{\varepsilon}_{n\times 1}$. (26)

   a. Show that $\dfrac{\left(\hat{\beta}-\beta\right)'\left(X'X\right)\left(\hat{\beta}-\beta\right)}{\sigma^2} \sim \chi_p^2$.

   b. Show that $\hat{\beta}$ is independent of s^2.

2. For Model given in question 1. (24)
   a. Define estimable $\boldsymbol{\ell}'\boldsymbol{\beta}$ where $\boldsymbol{\ell}'$ is a vector. What is the dimension of $\boldsymbol{\ell}'$?
   b. Define BLUE for $\boldsymbol{\ell}'\boldsymbol{\beta}$.
   c. State Gauss-Markov Theorem.

3. For model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, assume that $\boldsymbol{\varepsilon} \sim \mathbf{N}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$ The statistical hypothesis tests in linear regression model is formed as (26)

   $$\mathbf{H_o}: \mathbf{C}\boldsymbol{\beta} = \boldsymbol{\gamma} \qquad \mathbf{H_a}: \mathbf{C}\boldsymbol{\beta} \neq \boldsymbol{\gamma}$$

   where m is the rank of **C,** a m×(k+1) matrix. Prove the following,
   a. Give the test statistic and the distribution of the statistic
   b. Prove the test statistic in part 3.a above has the F distribution with degrees of freedom _____ and _____.

4. For model 1 (24)
   a. List all the parameters and their Least Square Estimates (LSE).
   b. List all the Maximum Likelihood Estimates (MLE). Are these estimates unbiased?
   c. What are the relations of LSE and MLE?
   d. Is it necessary to require $y_1, \ y_2, \dots y_n$ to be independent? Explain.