STA 682 Qualify Exam Fall 2025 Student Name ______________

General Instruction: This is a closed exam with no textbooks, notes and electronic devices. The exam lasts 180 minutes. Answer all questions. Print your answer and number the pages and number the problems on provided exam papers. Write on one side only. Please use only black pens or pencils.

Notation of Multiple Linear Regression Model (1):

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad X = \begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1k} \\ 1 & x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & \vdots & & \vdots & \\ 1 & x_{n1} & x_{n2} & \dots & x_{nk} \end{bmatrix} \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$E(\boldsymbol{\varepsilon}) = \mathbf{0} \qquad \boldsymbol{\varepsilon} \sim \boldsymbol{N}(0, \sigma^2 \boldsymbol{I}) \qquad Cov(\boldsymbol{\varepsilon}) = \sigma^2 \boldsymbol{I}$$

1. For Model (1)
   a. Define estimable $\boldsymbol{\ell}'\boldsymbol{\beta}$ where $\boldsymbol{\ell}'$ is a vector. What is the dimension of $\boldsymbol{\ell}'$? (3)
   b. Define BLUE for $\boldsymbol{\ell}'\boldsymbol{\beta}$. (3)
   c. State Gauss-Markov Theorem. (5)
   d. Prove Gauss-Markov Theorem. (9)

2. Given Model (1) Prove the following:

   a. $\boldsymbol{b} = \widehat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} \sim \mathrm{N}(\boldsymbol{\beta}, \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1})$ using moment generating function method. (6)
   b. $\frac{(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta})'(\boldsymbol{X}'\boldsymbol{X})(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta})}{\sigma^2} \sim \chi^2_p$. (6)
   c. $\hat{\boldsymbol{\beta}}$ and $\mathbf{S}^2$ are independent. (6)
   d. $\frac{(n-p)\boldsymbol{S}^2}{\sigma^2} \sim \chi^2_{n-p}$. (6)
   e. $\bar{y}$ and $\sum_{i=1}^{n}(y_i - \bar{y})^2$ are independent. (6)

3. For Model (1): $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, assume that $\boldsymbol{\varepsilon} \sim \boldsymbol{N}(0, \sigma^2 \boldsymbol{I})$ The statistical hypothesis tests in linear regression model is formed as

   $$\mathbf{H_o}: \mathbf{C}\boldsymbol{\beta} = \boldsymbol{\gamma} \qquad \mathbf{H_a}: \mathbf{C}\boldsymbol{\beta} \neq \boldsymbol{\gamma}$$

   where m is the rank of **C,** a m×(k+1) matrix.

   a. Give the test statistic and its distribution of the statistic (10)
   b. Prove that the test statistic in part 3.a. has the $F \sim F_{m,n-p}$ distribution. (16)

4. For Model (1)
   a. List all the parameters and their Least Square Estimates (LSE). (5)
   b. Derive all the Maximum Likelihood Estimates (MLE). Are these estimates unbiased? (11)
   c. What are the relations of LSE and MLE? (3)
   d. Is it necessary to require $y_1, y_2, \ldots y_n$ to be independent for deriving all the estimates above? Explain. (5)