General Instruction: This is a closed exam, with no any text, notes and electronic devices. The exam lasts 180 minutes. Answer all questions. Print your answer, your name, number the pages and number the problems on provided exam papers. Write on one side only. Please use only black pens or pencils.

Notation of Multiple Linear Regression Model (1): $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$ where

$$\boldsymbol{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \qquad \boldsymbol{X} = \begin{pmatrix} 1 & x_{11} & \cdots & x_{1k} \\ 1 & x_{21} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{nk} \end{pmatrix} \qquad \boldsymbol{\beta} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$$E(\boldsymbol{\varepsilon}) = \boldsymbol{0} \qquad \boldsymbol{\varepsilon} = N(\boldsymbol{0}, \sigma^2 \boldsymbol{I}) \qquad Cov(\boldsymbol{\varepsilon}) = \sigma^2 \boldsymbol{I}$$

1. For Model (1) (15)
   a. Define estimable $\boldsymbol{\ell}'\boldsymbol{\beta}$ where $\boldsymbol{\ell}'$ is a vector. What is the dimension of $\boldsymbol{\ell}'$?
   b. Define BLUE for $\boldsymbol{\ell}'\boldsymbol{\beta}$.
   c. State Gauss-Markov Theorem.
   d. Prove Gauss-Markov Theorem.

2. Given the model, $\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon} \sim \boldsymbol{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$, $\boldsymbol{Y}_{n\times 1}$, $\boldsymbol{X}_{n\times(k+1)}$, $\boldsymbol{\beta}_{(k+1)\times 1}$, $p=k+1$ and $\boldsymbol{\varepsilon}_{n\times 1}$. Prove the following: (20)
   a. $\boldsymbol{b} = \hat{\boldsymbol{\beta}} = (\boldsymbol{X'X})^{-1}\boldsymbol{X'Y} \sim N\left(\boldsymbol{\beta}, \sigma^2 (\boldsymbol{X'X})^{-1}\right)$ using moment generating function method.
   b. $\dfrac{(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})'(\boldsymbol{X'X})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})}{\sigma^2} \sim \chi^2_p$.
   c. $\hat{\boldsymbol{\beta}}$ and s^2 are independent
   d. $(n-p)s^2/\sigma^2 \sim \chi^2_{n-p}$.
   e. $\bar{Y}$ and $\sum_{i=1}^{n}(y_i - \bar{y})^2$ are independent.

3. For model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ with following partitions: (20)

   $\boldsymbol{Y} = \left(\boldsymbol{Y}^{(1)}\ \boldsymbol{Y}^{(2)}\right)$, $\boldsymbol{X} = \left(\boldsymbol{X}^{(1)}\ \boldsymbol{X}^{2}\right)$, $\boldsymbol{\beta}' = \left(\boldsymbol{\beta}^{(1)'}\ \boldsymbol{\beta}^{(2)'}\right)$, $\boldsymbol{\beta}^{(1)'} = (\beta_0, \beta_1 \ldots \beta_r)$, and $\boldsymbol{\beta}^{(2)'} = (\beta_{r+1}, \ldots \beta_p)$

   a. For test $\mathbf{H_o}: \boldsymbol{\beta}^{(2)} = \mathbf{0}$ $\quad \mathbf{H_a}: \boldsymbol{\beta}^{(2)} \neq \mathbf{0}$ find the $\boldsymbol{C}, \boldsymbol{\gamma}$ and m in the following test statistic.

$$\frac{(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{\gamma})'\left[\boldsymbol{C}(\boldsymbol{X'X})^{-1}\boldsymbol{C}'\right]^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{\gamma})}{\boldsymbol{y}'\left[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X'X})^{-1}\boldsymbol{X}'\right]\boldsymbol{y}}\left(\frac{n-k-1}{m}\right) \sim F_{m,\,n-p}$$

b. Given that

$$\left(y - X\hat{\beta}_H\right)'\left(y - X\hat{\beta}_H\right) - \left(y - X\hat{\beta}\right)'\left(y - X\hat{\beta}\right) = \left(X\hat{\beta} - X\hat{\beta}_H\right)'\left(X\hat{\beta} - X\hat{\beta}_H\right)$$

Prove

$$\left(X\hat{\beta} - X\hat{\beta}_H\right)'\left(X\hat{\beta} - X\hat{\beta}_H\right) = \left(C\hat{\beta} - \gamma\right)'\left[C\left(X'X\right)^{-1}C'\right]^{-1}\left(C\hat{\beta} - \gamma\right)$$

c. Give the SSE_F and SSE_R in their matrix form, and prove that

$$\frac{\left(C\hat{\beta} - \gamma\right)'\left[C\left(X'X\right)^{-1}C'\right]^{-1}\left(C\hat{\beta} - \gamma\right)}{y'\left[I - X\left(X'X\right)^{-1}X'\right]y}\left(\frac{n\text{-}k\text{-}1}{m}\right) = \frac{SSE_R - SSE_F}{SSE_F}\left(\frac{n-k-1}{m}\right) \sim F_{m,\, n-p}$$

e. Is the test a Maximum Likelihood Ratio test? What is the significance for it to be a Maximum Likelihood Ratio test?

4. For Model (1) (15)
   a. List all the parameters and their Least Square Estimates (LSE).
   b. List all the Maximum Likelihood Estimates (MLE). Are these estimates unbiased?
   c. What are the relations of LSE and MLE?
   d. Is it necessary to require $y_1,\ y_2, \dots y_n$ to be independent?

5. For Model (1) Let $\boldsymbol{X}_{(i)}$ be the matrix $\boldsymbol{X}$ with the ith row removed, (10)
   and $\boldsymbol{x_\ell} = (1 \quad x_{\ell 1} \quad \cdots \quad x_{\ell k})' \quad \ell = 1, 2, \dots n$ so that

$$\sum_{\boldsymbol{\ell=1}}^{\boldsymbol{n}} \boldsymbol{x_\ell x'_\ell} - \boldsymbol{x}_i\boldsymbol{x}'_i = \boldsymbol{X'X} - \boldsymbol{x}_i\boldsymbol{x}'_i$$

   Prove

$$\boldsymbol{b} - \boldsymbol{b}_{(i)} = \frac{(\boldsymbol{X'X})^{-\mathbf{1}}\boldsymbol{x}_i e_i}{\mathbf{1} - h_{ii}}$$

6. For Model (1) (20)
   a. Define the Studentized Residuals $e_i^* \quad i = 1, 2, \dots n$.
   b. Show that Studentized Residual in part a. has Student's t distribution with the degrees of freedom *n-k-2.*